Script to the lecture

Fundamentals of Electrical Engineering

For Students of the Bachelor Courses

- Elektrotechnik Jülich ET
- Electrical Engineering ET-AOS
- Physikingenieurwesen PI
- Physical Engineering PI-AOS
- Wirtschaftsingenieurwesen WI

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Fundamentals of electrical engineering

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(from the literature given some figures are used in this script)

1. Fundamentals

1.1 Physical units

Based on the standard DIN 1301, only **SI-units** should be used in sciences and engineering. This ensures that the technical items can be understood even through the language barrier.

(SI = Systeme International d' Unites; International System of Units)

All SI-units can be derived from the seven basic SI-units

physical dimension	**name of the SI-basic-unit**	**symbol of the SI-basic-unit**
length *l*	meter	m
mass *m*	kilogram	kg
time *t*	second	s
current *I*	ampere	A
thermodynamic temperature *T*	Kelvin	K *
substance *N*	mol	mol
light intensity *J*	candela	cd

[1)] 0° C = 273,15 K; for temperature [ΔT]=K; always !!!

Examples of additional SI-units:

Unit of force:	Newton	$1\,N = 1\,\frac{kg \cdot m}{s^2}$
Unit of energy (work):	Joule	$1\,J = 1\,N \cdot m = 1\,W \cdot s = 1\,\frac{kg \cdot m^2}{s^2}$
Unit of voltage:	Volt	$1\,W \cdot s = 1\,V \cdot A \cdot s \;\rightarrow 1\,V = 1\,\frac{kg \cdot m^2}{A \cdot s^3}$

Examples of units, which are not belonging to the SI-unit-system:

kp = 9.81 N cal = 4.187 J PS = 0.735 kW hp = 0.740 kW Torr = 1.33 hPa

Besides SI-units, the following units are also common:

1 min = 60 s $1° = \frac{\pi}{180} rad$

1 h = 3 600 s

1 d = 86 400 s 1 bar = 10^5 Pa

1 l = 10^{-3} m^3 1 eV = 0,1602*10^{-18} J

1 t = 10^3 kg 1 var = 1 VA

Decimal numbers

In engineering and sciences decimal multiples and dividends of the basic units are common.

Exception: Multiples of the time unit, the second.

1 d = 24 h 1 h = 60 min 1 min = 60 s

Multiple	Name	Symbol	Dividend	Name	Symbol
10^{24}	Yotta	Y	10^{-1}	Deci	d
10^{21}	Zeta	Z	10^{-2}	Centi	c
10^{18}	Exa	E	10^{-3}	Milli	m
10^{15}	Peta	P	10^{-6}	Micro	µ
10^{12}	Tera	T	10^{-9}	Nano	n
10^{9}	Giga	G	10^{-12}	Pico	p
10^{6}	Mega	M	10^{-15}	Femto	f
10^{3}	Kilo	k	10^{-18}	Atto	a
10^{2}	Hecto	h	10^{-21}	Zepto	z
10^{1}	Deca	da	10^{-24}	Yocto	y

Annotations:: German: 1 Milliarde = English: 1 billion
Please note the different manners of writing 1.200 and 1,200 in German and English equations !

1.2 Quantity equations

A physical quantity is the product of a number and a unit.

Example: gravity $g = 9{,}81\frac{m}{s^2} \quad (g \neq 9{,}81)$

Quantity equations always include physical quantities according to the definition above.

Example of a quantity equation:

$$P = \omega \cdot M$$

ω = angular velocity, for example $\omega = 100\frac{1}{s} = 100\frac{rad}{s} = 0{,}1\frac{rad}{ms} = \frac{5{,}73°}{ms}$

M = torque, for example M = 10 Nm

P = power $P = 100\frac{1}{s} \cdot 10Nm = 1\,000\,\frac{Nm}{s} = 1\,000\,\frac{Ws}{s} = 1kW$

It often makes sense to label the individual quantities in a quantity equation with certain units (for example: data evaluation, table construction, graphical illustration).

The quantity equation then becomes a specific quantity equation.

Example: The aim is an equation in which the power is given in kW as a function of the rounds in min $^{-1}$ and the torque in Nm.

$P = \omega \cdot M$ For the rotary motion, it is uncommon to give the angular velocity.

$\omega = 2\pi \cdot n$ In applied science the rotational numbers are often stated in rounds per minute (min^{-1})

$P = 2\pi \cdot n \cdot M$ ← generally valid relation

Every physical dimension is divided by the assigned unit.
Of course the equation must remain correct!

$$\frac{P}{kW} = 2\pi \cdot \frac{n}{\min^{-1}} \cdot \frac{M}{Nm} \cdot \frac{1}{kW} \cdot \min^{-1} \cdot Nm$$

$$\frac{P}{kW} = 2\pi \cdot \frac{Nm}{1\,000\,W \cdot \min} \cdot \frac{n}{\min^{-1}} \cdot \frac{M}{Nm}$$

↑ ↑ ↑ ↑ Quantities without dimensions

$$\frac{P}{kW} = 2\pi \cdot \frac{Ws}{1\,000\,W \cdot 60\,s} \cdot \frac{n}{\min^{-1}} \cdot \frac{M}{Nm}$$

$$\frac{P}{kW} = 0{,}1047 \cdot 10^{-3} \cdot \frac{n}{\min^{-1}} \cdot \frac{M}{Nm}$$

For Example: n = 3000 min^{-1}; M = 10 Nm

$$\frac{P}{kW} = 0{,}1047 \cdot 10^{-3} \cdot \frac{3\,000\ \min^{-1}}{\min^{-1}} \cdot \frac{10\ Nm}{Nm} = 3{,}141 \rightarrow \underline{P = 3{,}141\ kW}$$

1.3 Character of electricity

A description of a well-known phenomenon (see also chapter 2.1)
“If two amber rods are rubbed against each other, they repel each other. The same happens to two glass rods. However, if an amber rod is rubbed against a glass rod, they attract each other.”

- Logical conclusion: There are obviously two different kinds of electrical charges.

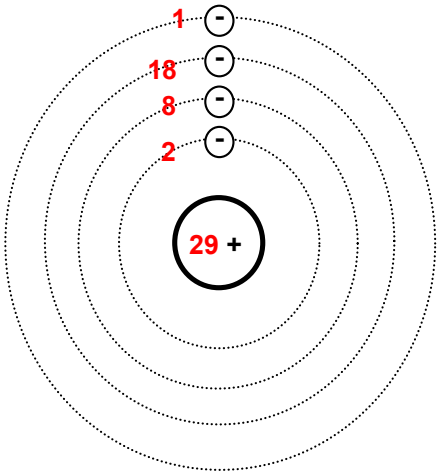
- In Greek, amber means “το ηλεκτρον“, from which the word electricity was derived.

- The polarity was determined by the definition that glass was positively charged and that the amber was negatively charged (Chr. Lichtenberg, 1777).

1.3.1 Bohr's modell

(Niels Bohr, 1913)

Example: copper



In the atomic nucleus:

Protons: 29
Neutrons: 34 (Cu 63)
36 (Cu 65)
Isotope

(relative atomic mass: 63,54)

On the electron orbit:

29 electrons

Comparisons of size ranges:

∅ atom	≈ 10^{-10} m	∅ Pluto Orbit	$12*10^{9}$ km
∅ atomic nucleus	≈ 10^{-15} m	∅ Earth Orbit	$300*10^{6}$ km
(estimated values)		∅ Sun	$1{,}4*10^{6}$ km

	mass	charge
Proton	$1{,}672*10^{-27}$ kg	+ $0{,}1602*10^{-18}$ As
Neutron	$1{,}675*10^{-27}$ kg	± 0
Electron	$0{,}911*10^{-30}$ kg	- $0{,}1602*10^{-18}$ As

$$\frac{m_+}{m_-} = 1835$$

Comparison: $\frac{mass\ of\ the\ sun}{mass\ of\ the\ earth} = 330\,000$ $\frac{mass\ of\ the\ sun}{mass\ of\ all\ \ planets} = 743$

The nucleus is positively charged, the electrons are negatively charged: This is the result of the definition of polarity from the year 1777

The atomic structure shows whether a specific material has good or bad electrical conductivity.

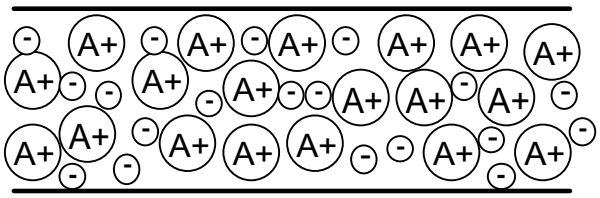
Example: copper:

There is only a very small gravitational force on the outer electron (the 29th). This is because of the relative larger distance to the atomic nucleus, and that the 29-times positively charged atomic nucleus is nearly completely compensated by the 28

electrons on the K-, L- and M- electron orbits. Therefore, the outer electron can very easily move away from the nucleus:

→ **Conduction electron, free electron**

1.3.2 Electrical voltage and current



Conduction electrons in a copper wire are not linked to specific atomic nuclei anymore. They can move relatively freely among the atomic nuclei (or better: among positive ions), like in a liquid or in a gas or in porous material.

⇒ „electron-gas”

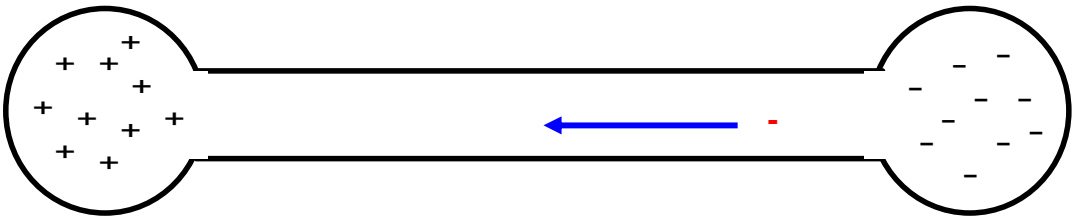
Metallic conductivity:

Normal state: chaotic movement of the conduction electrons without a general direction

In the case of a current flow: now a drifting movement in a general direction is identifiable

flow of an electric current = transport of electric charges

Reason for the drifting movement:

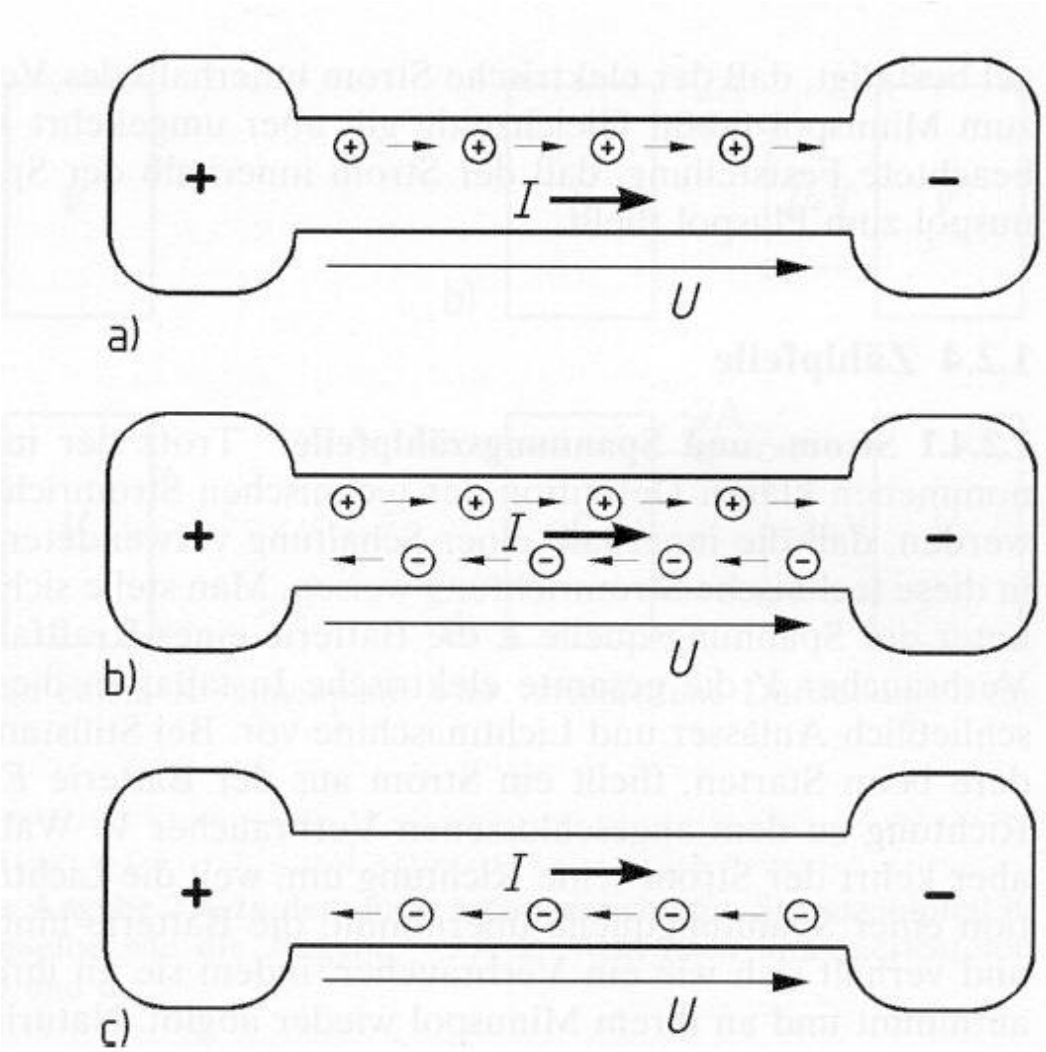


Area with a deficiency of electrons (few electrons)

Area with an abundance of electrons (many electrons)

In the past, this current flow was described as a “physical current direction”, a term, which should not be used anymore. The current flow direction is determined to be from positive to negative, which used to be referred to as “technical current direction.”

As you can see, all three cases described below lead to the same current flow *I*. In an electric circuit it apparently does not matter if one supposes that positive charges move from left to right (clockwise), or that negative charges move from right to left (counterclockwise), or both [1].



high potential low potential

The general model used now presents the motion of positive charges (= a positive current *I*). Although it delivers the correct results, the model based on the motion of positive charges is not absolutely correct for metallic conductivity.

A similar comparison could be ‘decreasing in heat = increasing in cold’.

Electric current strength = electric charge transported per unit of time

Current strength	I	Unit [I]	= A [Ampere]
Potential	φ	Unit [φ]	= V [Volt]
Voltage	U	Unit [U]	= V

This process is comparable to an exchange of a liquid or gas through a tube which connects two tanks under different pressures:

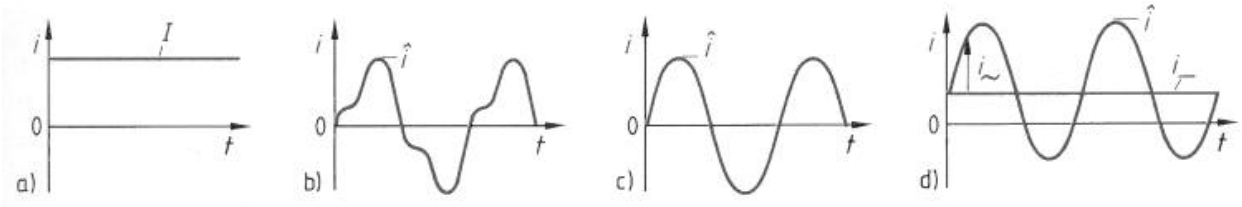
Flow rate	$\Delta V/\Delta t$	Unit $[\Delta V/\Delta t]$	= m^3/s	$\approx I$
Pressure	p	Unit $[p]$	= Pa	$\approx \varphi$
Pressure difference	Δp	Unit $[\Delta p]$	= Pa	$\approx U$

Types of electricity

It is common to classify types of electricity in:

- direct current – **DC** with constant, unchanging flow direction.
- alternating current – **AC** with not constant, changing flow direction.

Naturally, more combinations are possible (see examples) [1].

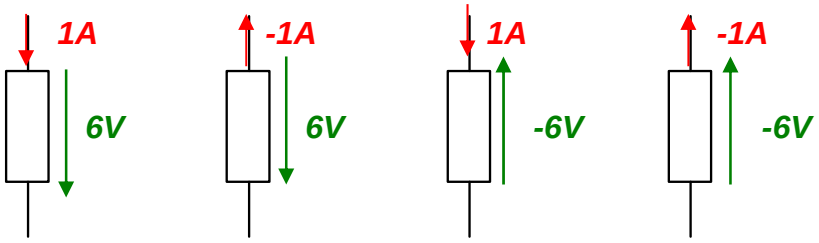


a) direct current I
b) general periodic alternating current i with peak value $\hat{\imath}$
c) sinusoidal alternating current i with peak value $\hat{\imath}$
d) "mixed" current: direct current i_- with superposed sinusoidal alternating current $i_\sim$

1.3.3 Directional arrows for currents and voltages

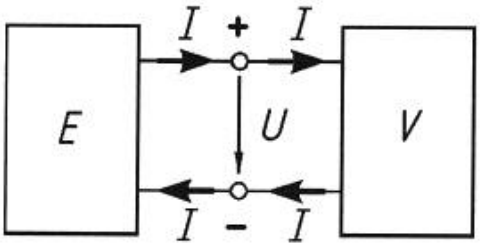
The value next to the voltage arrow indicates how much the potential at the end of the arrow is higher than at the arrow's tip.

The value next to the current arrow indicates how much positive charge per unit of time is transported in the arrow direction.

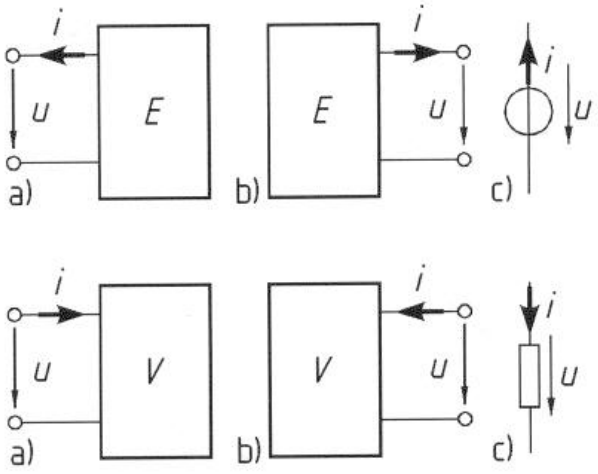


Four different descriptions of one condition

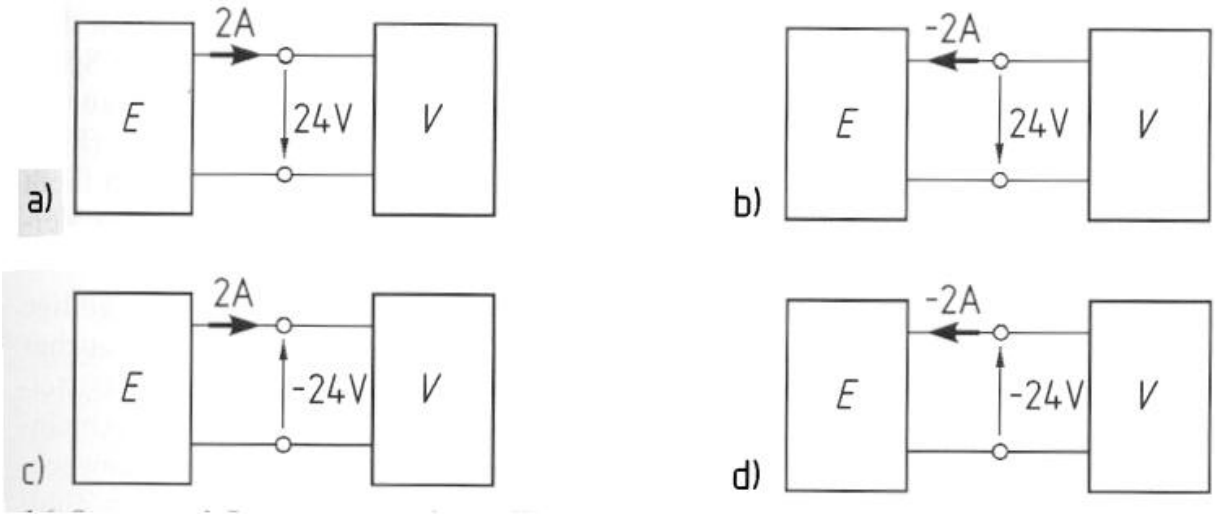
In case of a closed electric circuit consisting of a source (generator, E) and a load (consumer, V), voltage and current have the directions shown besides, if the so-called consumer-related system of directional arrows is used [1].



This indicates that in case of an element consuming physical power (V) current and voltage have the same direction (hence the power is positive). Consequently, in case of a source of electric energy (E), voltage and current must have opposite directions (hence the power is negative).



For a closed electric circuit with source E and load V, there are again four possible descriptions of the same situation:



1.4 Physical fields

1.4.1 General field terms

Although this lecture series only covers stationary electrical fields, a general description of time-varying physical fields shall also be introduced.

Field quantity:
Physical quantity which assumes a clear and distinct value for every point in a space.

Field:
Entirety of all values of a field quantity occurring within the space. The physical field describes a certain physical condition of the space. The space itself is the “carrier” of this field.

Scalar field:
The field quantity is a scalar (undirected) quantity.

Examples: $T(x,y)$ temperature
$\varphi(x,y)$ potential
$H(x,y)$ contour line from a mountain

Vector field:
The field quantity is a vectorial (directed) quantity.

Examples: $\vec{v}(x,y)$ speed
$\vec{J}(x,y)$ current density

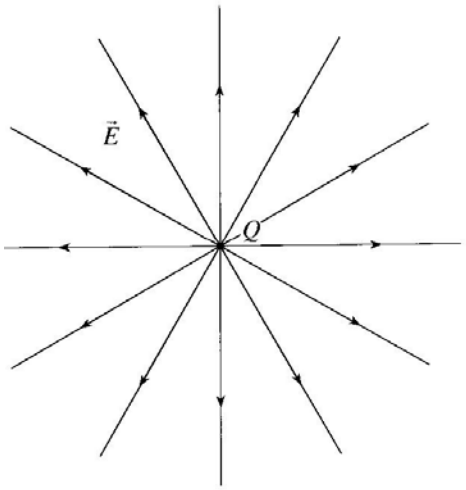
Attention: Current I and Voltage U are not field quantites because they cannot be correlated with a point in space. The potential φ, on the other hand, is a field quantity.

1.4.2 Vector fields

Because every point in space correlates with a vector a selection for the depiction must be decided upon (e.g. equidistant grid).

Field lines: Three dimensional space curves (lines) which run tangential to the corresponding field vector correlating with each point in space.

Example: field lines of a point charge



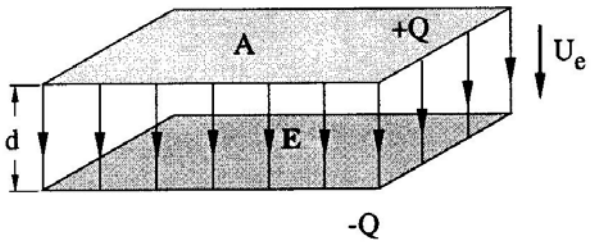
$$\vec{E}(\vec{r}) = \frac{Q}{4 \cdot \pi \cdot \varepsilon \cdot r^2} \cdot \vec{e}_r$$

The field lines radiate out from a central point.

Homogenous and Inhomogeneous fields

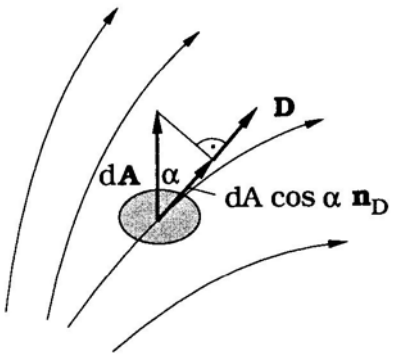
Physical fields can be distinguished by their spatial arrangement [5].

Homogenous fields:
Quantity and direction are independent from position.



Example: Electrical field of a plate capacitor

Inhomogeneous fields
Quantity and direction are dependent upon position.



Example: Electrical flux density

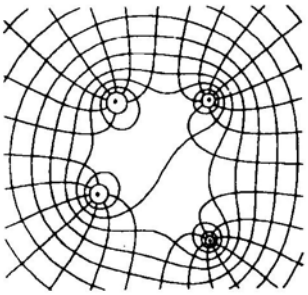
Fundamental types of vector fields

Field lines can be closed (vortex) or they can begin at definite points (sources) and end at other points (sinks). All vector fields can be understood as superposition of source fields and vortex fields.

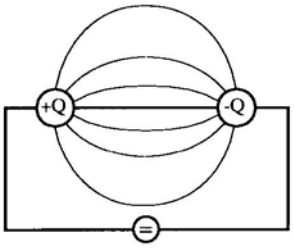
Pure **source fields** are vortex free fields, it means that

$$\oint_C \vec{F} \cdot d\vec{s} = 0$$

C: any closed line curve



a)



b)

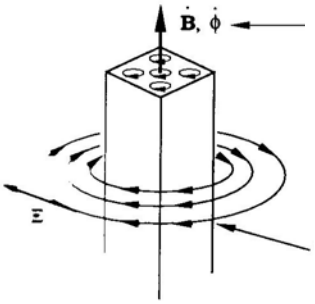
Example: Electrical fields

a) space / volume charge field (4 charges) without boundary conditions
b) two electrodes with different charges

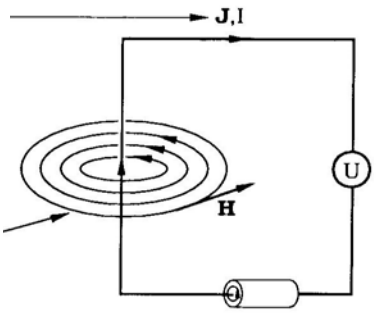
Pure **vortex fields** are source free fields, it means that

$$\oint_A \vec{F} \cdot d\vec{A} = 0$$

A: any closed surface area



a)



b)

Examples:

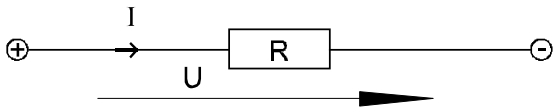
a) Electrical vortex field with time-varying magnetic field (induction)
b) Magnetic vortex field around an electrified conductor

2. Analysis of Direct Current Circuits

2.1 Ohm's Law

The electrons have to work through the atomic lattice of the conductive material, they also have to overcome a certain resistance. Therefore the speed of the electrons is very low (< 1 mm/s).

Physical quantity: **electric resistance**



$I \sim U$ for R = const.

$I \sim \frac{1}{R}$ for U = const.

$I \sim \frac{U}{R}$

Due to the definition of the unit for the electric resistance, the proportional factor is simply 1.

$$\boxed{I = \frac{U}{R}} \qquad \boxed{U = R \cdot I} \qquad \boxed{R = \frac{U}{I}} \qquad Unit[R] = \frac{[U]}{[I]} = \frac{V}{A} = \Omega \ (Ohm)$$

The ohmic resistance indicates the difficulty for the current to flow through a two-terminal network.

How easily the current flows through a two-terminal network is described by the physical quantity: **electric conductance**.

$$\boxed{I = G \cdot U} \qquad \boxed{U = \frac{I}{G}} \qquad \boxed{G = \frac{I}{U}} \qquad Unit[G] = \frac{[I]}{[U]} = \frac{A}{V} = \frac{1}{\Omega} = S \ (Siemens) = MHO$$

$$\boxed{G = \frac{1}{R}}$$

Ohm's law states that the current and voltage are proportional to each other.

$$I \sim U \qquad R \neq f(U, I) \qquad G \neq f(U, I)$$

2.1.1 Electric resistivity and conductivity

Resistance and conductance of a long and straight conductor (wire)

$$R \sim \frac{l}{A} \qquad G \sim \frac{A}{l}$$

and depending on material $\Rightarrow$

$$\boxed{\begin{aligned} R &= \rho \cdot \frac{l}{A} = \frac{l}{\kappa \cdot A} \\ G &= \kappa \cdot \frac{A}{l} = \frac{A}{\rho \cdot l} \end{aligned}}$$

ρ = (electric) resistivity

κ = (electric) conductivity

Unit $[\rho] = \Omega \cdot m = 10^6 \frac{\Omega \cdot mm^2}{m}$

Unit $[\kappa] = \frac{S}{m} = 10^{-6} \frac{Sm}{mm^2}$

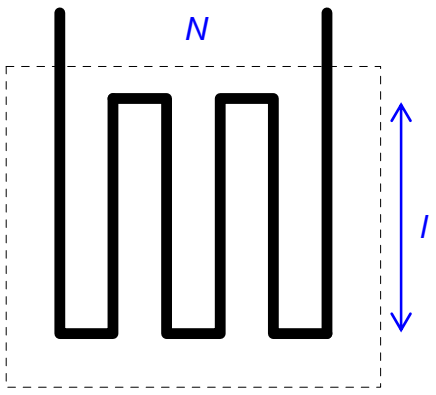
	$\frac{\rho}{\frac{\Omega mm^2}{m}}$	$\frac{\rho}{\Omega m}$	$\frac{\kappa}{\frac{Sm}{mm^2}}$	$\frac{\kappa}{\frac{S}{m}}$
Cu	0,018	$18 \cdot 10^{-9}$	56	$56 \cdot 10^6$
Al	0,029	$29 \cdot 10^{-9}$	35	$35 \cdot 10^6$
Ag	0,016	$16 \cdot 10^{-9}$	62,5	$62,5 \cdot 10^6$
Manganin	0,42	$420 \cdot 10^{-9}$	2,4	$2,4 \cdot 10^6$

2.1.2 Mechanically dependent resistances

Since $R = \rho \cdot {}^{l}\!/_{A}$ mechanical deformations may influence the resistance value:

- without remarkable influence: bending of a thin wire
- with remarkable influence: crimping, compressing, stretching (change of *l* or *A*)

Technically important: strain gauge (in german: Dehnungsmessstreifen DMS)



strain gauge

Resistor wire is fixed (bonded) in loops or in zigzag routing on a plastic material.

This strain gauge then is fixed on a test specimen.

Stretching, etc. leads to resistance change of the strain gauge.

unstretched: $R = \rho \cdot \frac{l_1}{A_1}$

wire length: $l_1 = N \cdot l$

wire volume: $V = l_1 \cdot A_1 = N \cdot l \cdot A_1$

stretched: $R + \Delta R = \rho \cdot \frac{l_2}{A_2}$

wire length: $l_2 = N \cdot (l + \Delta l)$

wire volume: $V = l_2 \cdot A_2 = N \cdot (l + \Delta l) \cdot A_2$

Assuming the wire volume remains constant*:

$$V = N \cdot l \cdot A_1 = N \cdot (l + \Delta l) \cdot A_2$$

$$\Rightarrow A_2 = \frac{l}{l + \Delta l} \cdot A_1$$

* Remark: This is a rough approximation; an exact calculation is possible with the poisson-number μ:
$\frac{\Delta r}{r} = \mu \cdot \frac{\Delta l}{l}$, r being the radius; for copper as an example μ = 0.35, whereas for a constant volume μ = 0.5 would be necessary.

$$R + \Delta R = \rho \cdot \frac{l_2}{A_2} = \rho \cdot \frac{N \cdot (l + \Delta l)}{\frac{l}{l + \Delta l} \cdot A_1} = \rho \cdot \frac{N \cdot (l + \Delta l)^2}{l \cdot A_1}$$

$$R + \Delta R = \rho \cdot \frac{N \cdot (l^2 + 2 \cdot l \cdot \Delta l + \Delta l^2)}{l \cdot A_1} \approx \rho \cdot \frac{N \cdot (l + 2 \cdot \Delta l)}{A_1}$$

$$R + \Delta R \approx \rho \cdot \frac{N \cdot l}{A_1} + \rho \cdot \frac{2 \cdot N \cdot \Delta l}{A_1}$$

$$\frac{\Delta R}{R} \approx \frac{\rho \cdot \frac{2 \cdot N \cdot \Delta l}{A_1}}{\rho \cdot \frac{N \cdot l}{A_1}} = \frac{2 \cdot \Delta l}{l}$$

$$\boxed{\frac{\Delta R}{R} = k \cdot \frac{\Delta l}{l}}$$

In reality *k* may be little different from *k* = 2 ⇒ calibration !

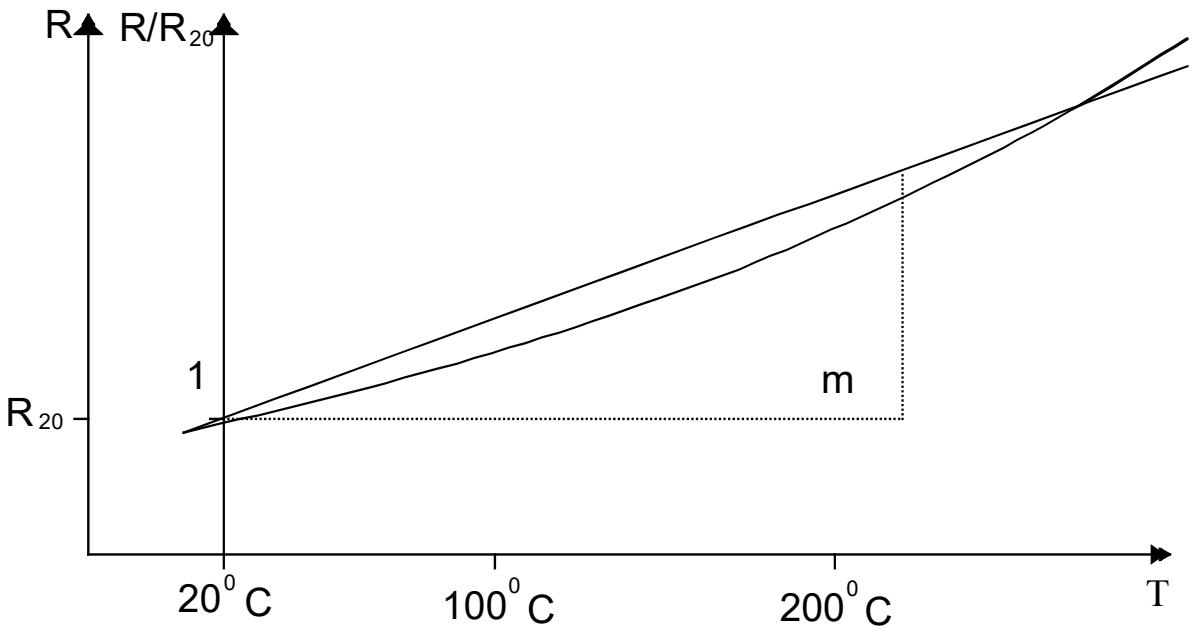
2.1.3 Temperature-dependence of resistances

Metallic conductor: Electrons are hindered due to the oscillation of the atomic lattice (oscillation increases when temperature increases)
→ Resistance increases caused by the temperature increase;
→ positive temperature coefficient of resistance.

Semiconductor: Charge charriers are hindered due to the oscillation of the atomic lattice as in case of metallic conductors.
However as energy is transported (heat) further electrons are able to leave the atomic torsos (valence band → conduction band). This leads to an increase of the number of charge carriers.

This effect dominates:
→ Resistance decreases caused by the temperature increase;
→ negative temperature coefficient of resistance.

The description *R(T)* is nonlinear, but in sections linearization is possible by approximation.



$$\frac{R}{R_{20}} = 1 + m \cdot \Delta T$$

slope $m = \alpha_{20}$ = temperature coefficient

$$\Delta T = T - 20°C$$

$$\boxed{R = R_{20} \cdot (1 + \alpha_{20} \cdot \Delta T)}$$

Linearization in the range of 20°C, therefore α_{20}
It is usually adequate in the range of 20°C up to approx. 200°C.

The following equation of the *R*(*T*) - curve has to be used, if the approximation should be useful for a larger temperature range or if the calculation should be more exact.

$$\boxed{R = R_{20} \cdot (1 + \alpha_{20} \cdot \Delta T + \beta_{20} \cdot \Delta T^2)}$$

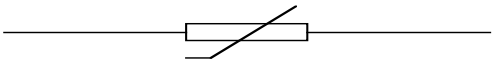
Approximation through a parabola:

Numerical values:

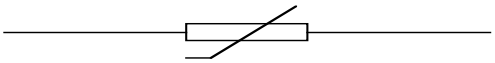
	$\frac{\alpha_{20}}{10^{-3}K^{-1}}$	$\frac{\beta_{20}}{10^{-6}K^{-2}}$
Silver	3,8	0,7
Copper	3,9	0,6
Aluminium	3,8	1,3
Magnesium	0,04	
Constantan	-0,004	
Graphite	-1,3	

Constantan: 53% Cu, 45% Ni, Mn, Al, Si (used for current shunts)

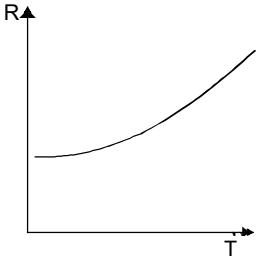
Special temperature-dependent characteristic of resistances:

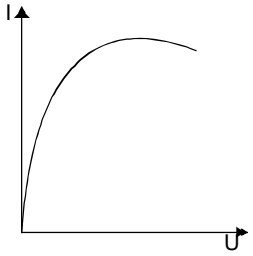


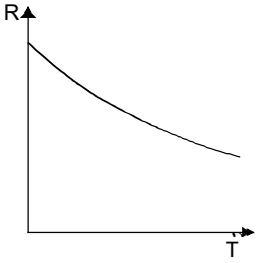
PTC – resistance / - thermistor
(positive temperature coefficient)

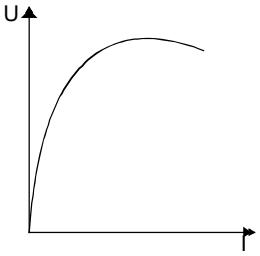


NTC – resistance / - thermistor
(negative temperature coefficient)





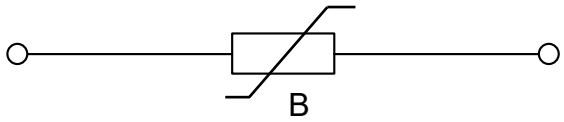




The curves are not isothermal. They apply to the stationary (no more varying) conditions for each achieved operating temperature.
(Please not also the interchanged axis in the I-U-diagrams)

2.1.4 Magnetically-dependent resistances

For some conductive materials (example: bismuth) and for semi-conductors the resistance is dependent on the magnetic field.



Technical applications: magnetic field sensors

Electrical charges are deflected by the magnetic field (see chapter: magnetic field) → detour → increased length → increased resistance

Equation based on experimental data: $\boxed{R = R_0 \cdot (1 + k \cdot B^2)}$ *B*: magnetic flux density

2.1.5 Optically-dependent resistances

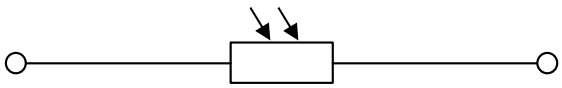
Internal photo effect:
Electrons which are at first bonded to the nucleus absorb the energy of a photon. Hence, they get more kinetic energy and can move away from the nucleus easily. Now they can act as conducting electrons (free electrons).

increased light intensity → increased number of electric charges (electrons)
→ decreased resistance

Differences in the spectral sensitivity:

- maximum in the visible range (0,38 μm ... 0,78 μm): cadmium sulfide (CdS), selenium (Se)
- maximum in the near infrared range: silicon (Si): 1,1 μm
  germanium (Ge): 1,7 μm
  lead sulphur (PbS): 2,8 μm

Photoresistors:



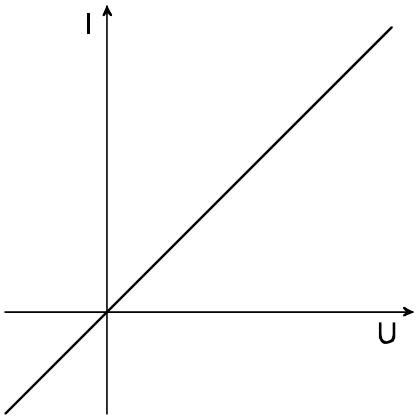
Material usually CdS, sometimes Se

2.1.6 Validity range (scope) of Ohm's law, nonlinear resistances

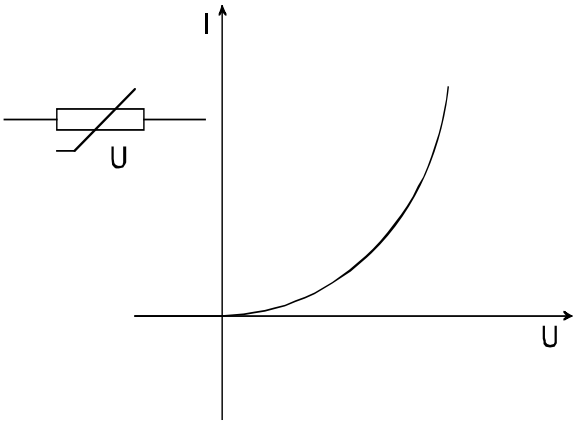
The equation $U = R * I$ is always valid.

However, it is only a formulation for Ohm's law if $R \neq f(U,I)$

Linear resistance:
Ohm's law valid



Nonlinear resistance:
Ohm's law invalid



Example: VDR (voltage dependent resistor, varistor)

Approximate description:

$$I/A = C \cdot \left(U/V\right)^n$$

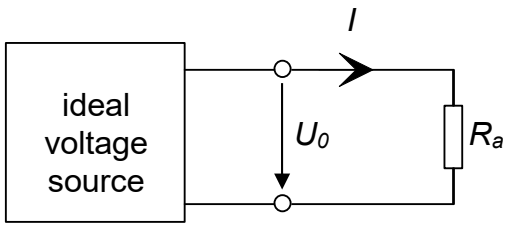
Ohm's law is not valid in many fields of semi-conductors and gas-discharges.

2.2 Equivalent circuits for practical voltage and current sources

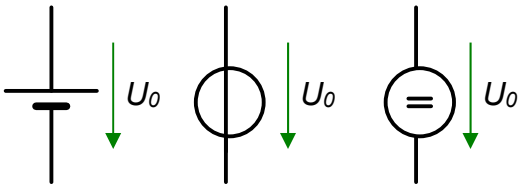
2.2.1 Ideal voltage sources

This is an ideal (fictitious) element in an electric circuit, an active two-terminal. Ideal voltage sources keep their terminals at a constant voltage U_0 under all operating conditions.

Ideal, fictitious



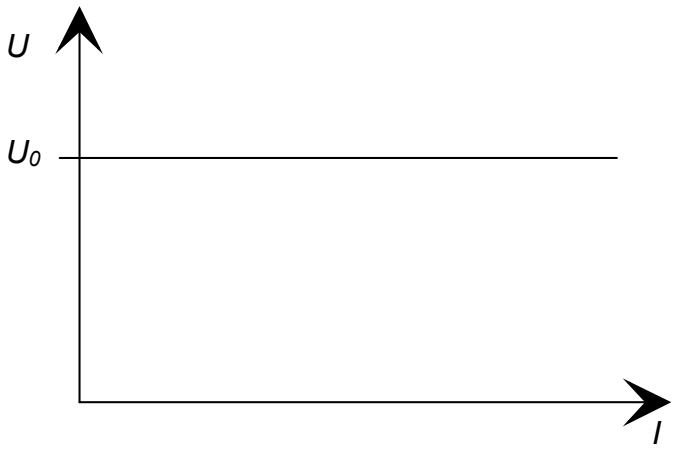
Symbols



derived from voltaic element

derived from generator

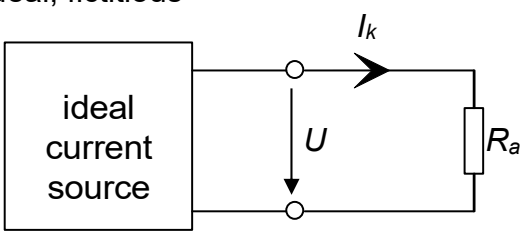
Curve, graph



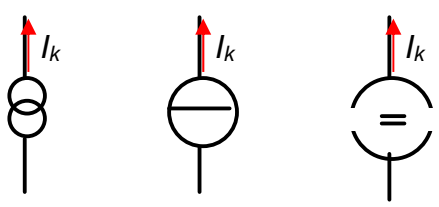
2.2.2 Ideal current source

They keep a constant current I_k at their terminals under all operating conditions.

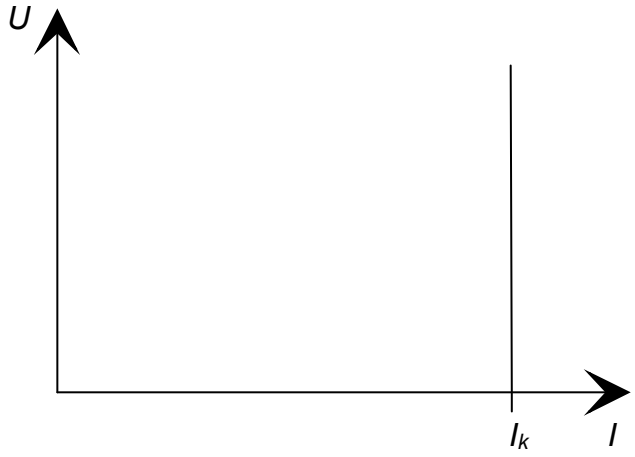
Ideal, fictitious



Symbols

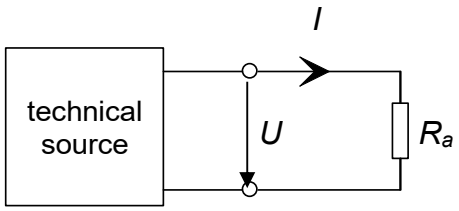


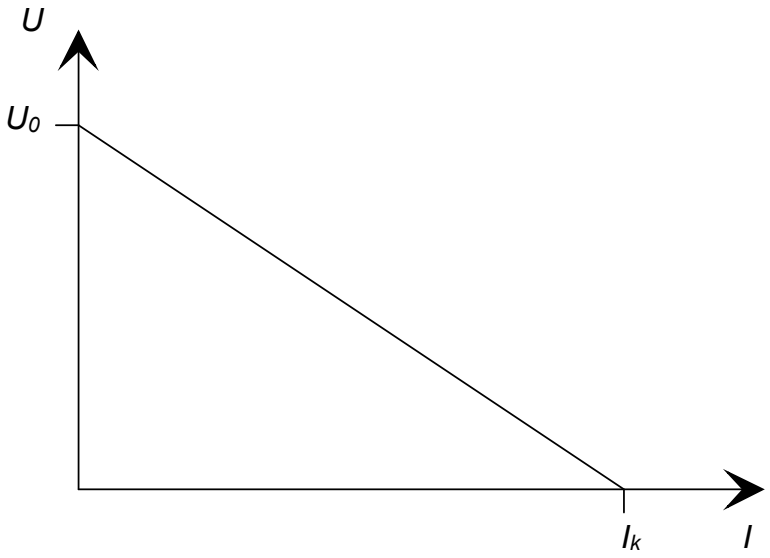
Curve, graph



2.2.3 Curves of technical (realistic) linear sources

As can be estimated easily, perfectly ideal elements are not possible in reality.
In praxis the measured curve *U(I)* for a technical source is a straight line (with a good approximation).



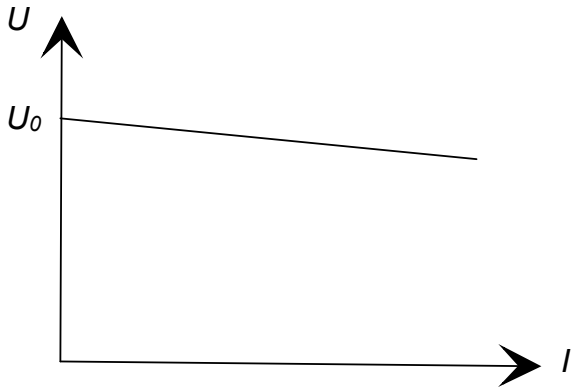


Useful descriptions:

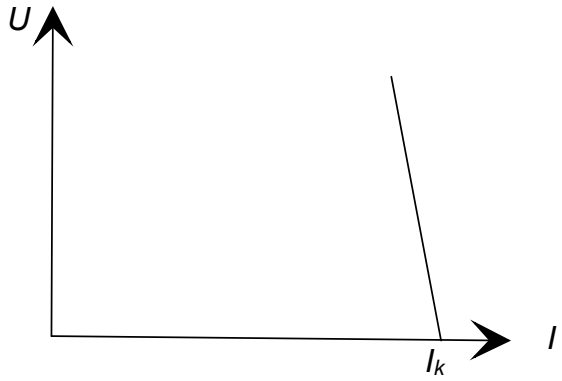
"Technical voltage source"
- for a graph with very low slope (nearly horizontal)

"Technical current source"
- for a graph with very high slope (nearly vertical)

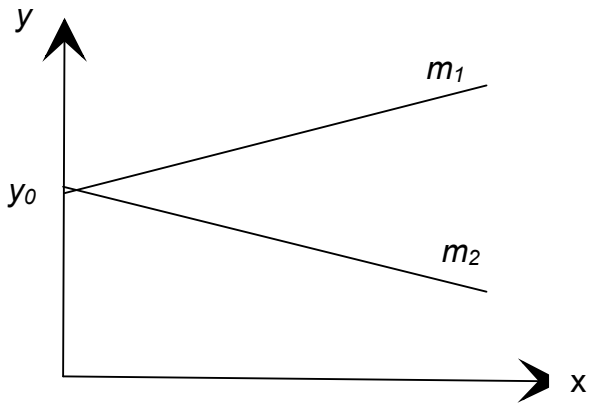
"Voltage stabilizer"

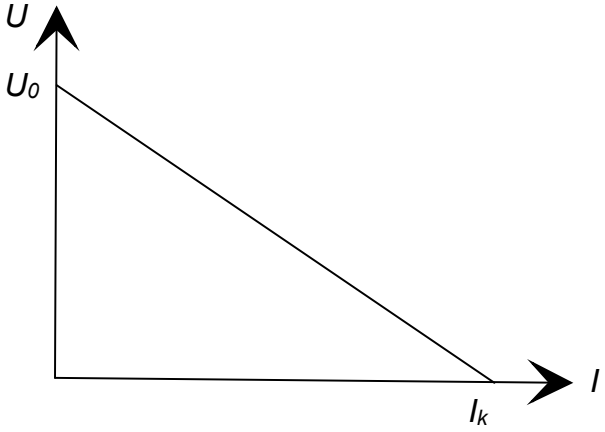


"Current stabilizer"



Analytical description of the graph:





Equation for a straight line:

$$y = y_0 + m \cdot x$$

$x \rightarrow I$
$y \rightarrow U$
$y_0 \rightarrow U_0$
$m \rightarrow R_i$

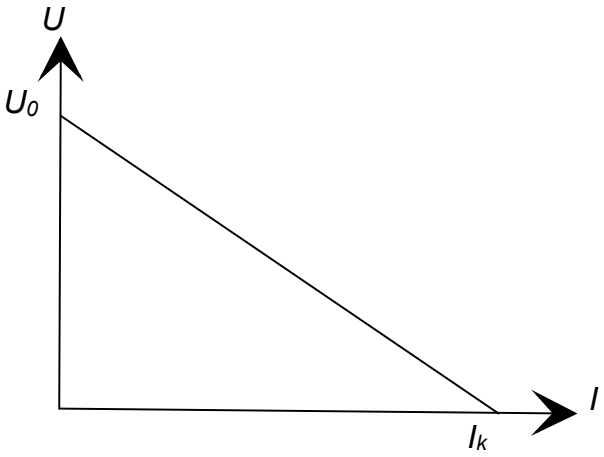
with: $m_1 > 0$
$m_2 < 0$

$$\boxed{U = U_0 - R_i \cdot I}$$

3 characteristic quantities

U_0 open-circuit voltage
I_k short-circuit current
R_i intrinsic (inner) resistance

Short-circuit point: $0 = U_0 - R_i \cdot I_k$ $\quad \boxed{U_0 = R_i \cdot I_k}$



It is also possible to get *I* by a transformation of the straight line's equation:

$$U = U_0 - R_i \cdot I$$

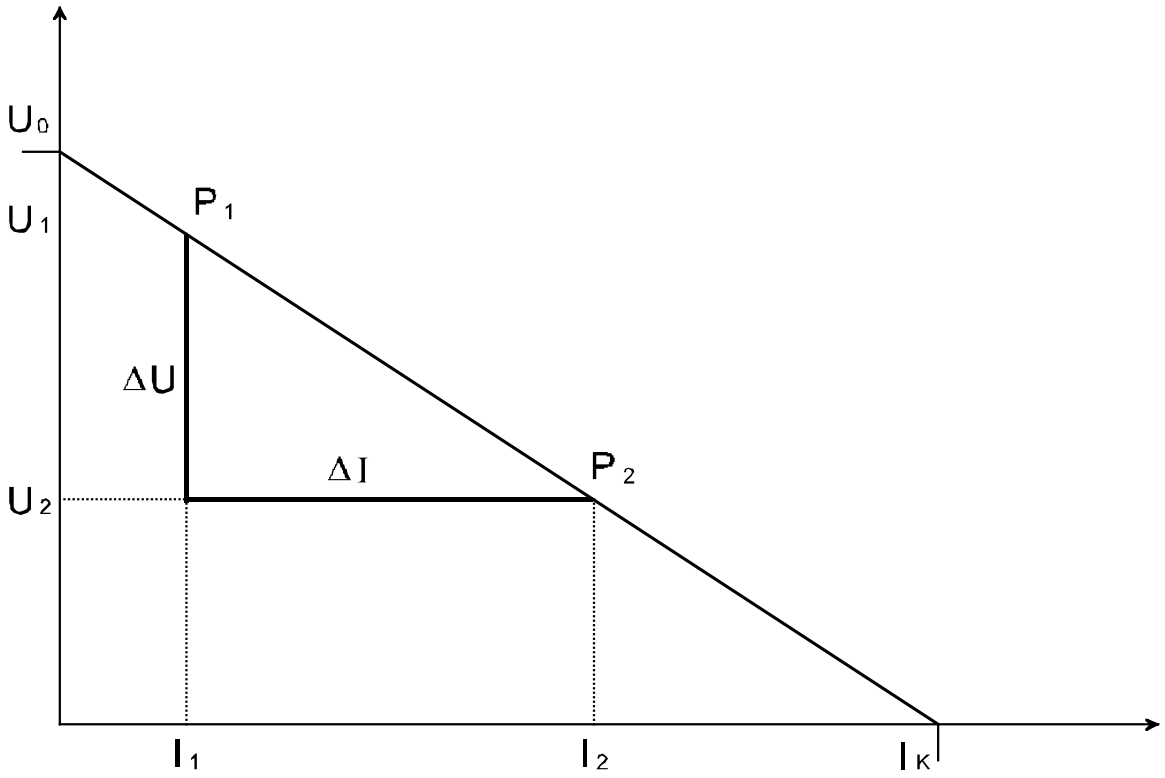
$$I = \frac{U_0 - U}{R_i} = \frac{U_0}{R_i} - \frac{U}{R_i}$$

$$\boxed{I = I_k - \frac{U}{R_i}}$$

Measurement of the characteristic quantities U_0, I_k, R_i:

Mostly the measurement of I_k is technically impossible; sometimes the measurement of U_0 is also difficult (e.g. thermoelement).

A straight line is unambiguously defined by two points. With 2 measured points P_1, P_2 the characteristic quantities can be determined.



Definition of R_i:

similar triangles $\quad \Rightarrow \frac{U_0}{I_K} = \frac{\Delta U}{\Delta I} \Rightarrow \underline{\underline{R_i = \frac{\Delta U}{\Delta I}}}$

ΔU and ΔI represent lines in the above diagram; therefore they are always positive.

Definition of U_0:

Both measuring points have to fulfil the equation $U = U_0 - R_i \cdot I$
(R_i is known in the meantime).

$$U_1 = U_0 - R_i \cdot I_1 \quad \text{or} \quad U_2 = U_0 - R_i \cdot I_2 \quad \Leftrightarrow$$
$$U_0 = U_1 + R_i \cdot I_1 \qquad U_0 = U_2 + R_i \cdot I_2$$

Of course, both equations give the same value for U_0:

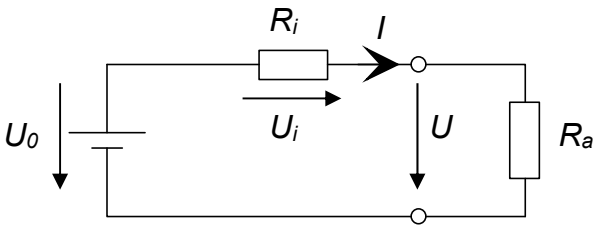
$$U_0 = U_1 + \frac{U_1 - U_2}{I_2 - I_1} \cdot I_1 \qquad U_0 = U_2 + \frac{U_1 - U_2}{I_2 - I_1} \cdot I_2$$
$$U_0 = \frac{U_1 \cdot I_2 - U_1 \cdot I_1 + U_1 \cdot I_1 - U_2 \cdot I_1}{I_2 - I_1} \qquad U_0 = \frac{U_2 \cdot I_2 - U_2 \cdot I_1 + U_1 \cdot I_2 - U_2 \cdot I_2}{I_2 - I_1}$$

$$U_0 = \frac{U_1 \cdot I_2 - U_2 \cdot I_1}{I_2 - I_1}$$

,

2.2.4 Equivalent circuit for voltage sources

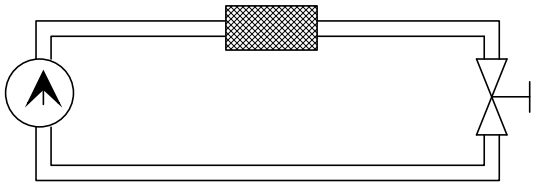
Equivalent circuit



$U_0 = U_i + U$
$U = U_0 - U_i \qquad U_i = R_i \cdot I$

$\boxed{U = U_0 - R_i \cdot I}$

Mechanical Analogy



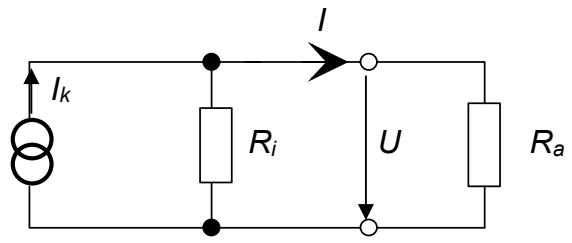
Ideal pump with Δp_0 = const. driving a turbine. Between the two devices the flow resistance has to be taken into account (pipes, cleaning filter).
$\Delta p_0 = \Delta p_i + \Delta p$

The equivalent circuit corresponds to the same relation as the technical source (see 2.2.3).

Ideal voltage source: Equivalent circuit for voltage source with $R_i = 0$.

2.2.5 Equivalent circuit for current sources

Equivalent circuit

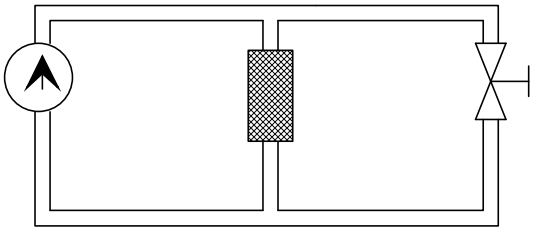


$$I_k = I_i + I$$

$$I = I_k - I_i \qquad I_i = \frac{U}{R_i}$$

$$\boxed{I = I_k - \frac{U}{R_i}}$$

Mechanical analogy



Ideal pump with V_p = const. (constant output volume of the pump) driving a turbine. Because of the characteristic of the valve a flow resistance has to be taken into account parallel to both devices.

$$V_p = V_i + V$$

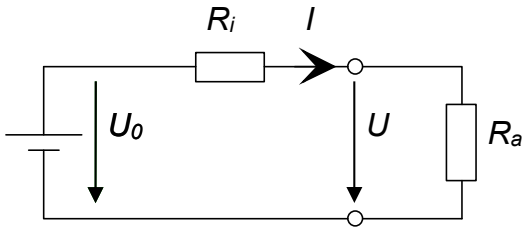
The equivalent circuit corresponds to the same relation as the technical source (see 2.2.3).

Ideal curent source: Equivalent circuit for voltage source with $R_i \rightarrow \infty$.

2.2.6 Equivalence of voltage and current sources

It is possible to describe a technical linear source as a technical voltage source or as a technical current source. Hence, dependent on the application, a transition of a technical voltage source into a technical current source and vice versa sometimes is necessary.

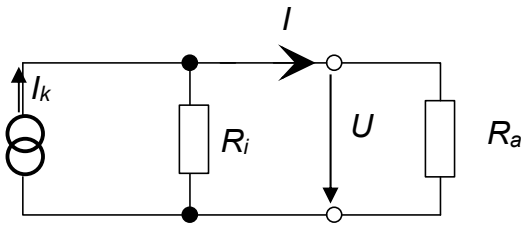
Technical voltage source



$$R_i = \frac{1}{G_i}$$

$$U_0 = R_i \cdot I_k = \frac{I_k}{G_i}$$

Technical current source



$$G_i = \frac{1}{R_i}$$

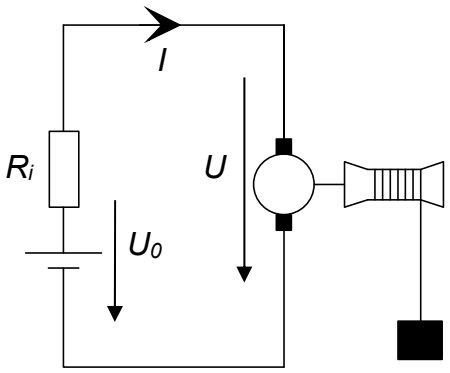
$$I_k = G_i \cdot U_0 = \frac{U_0}{R_i}$$

2.3 Electrical power and energy (work)

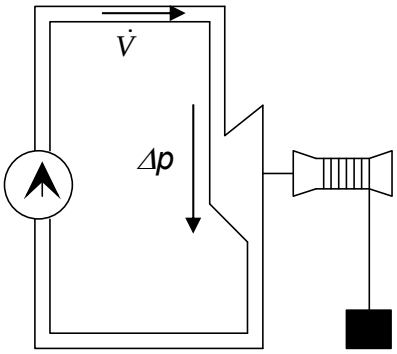
2.3.1 Calculation of power and energy

We assume we have the task to lift up a weight. This can be done either mechanically by using a turbine which is driven by a technical pump, or electrically with a motor getting the electrical energy from a technical voltage source.

Technical voltage source



Technical pump



Power at the motor

$P \sim I$

$P \sim U$

Power at the turbine

$P \sim \dot{V}$

$P \sim \Delta p$

If the work is carried out it is difficult to force water through the turbine: A pressure difference is built at the turbine which has to be overcome.

Due to the proper choice of the units, the proportionality factor is 1 in both cases:

$$\boxed{P = U \cdot I}$$

$$1W = 1V \cdot 1A \Leftrightarrow$$

$$\underline{\underline{1V = 1\frac{W}{A} = 1\frac{kg \cdot m^2}{A \cdot s^3}}}$$

$$\boxed{P = \Delta p \cdot \dot{V}}$$

$$1W = 1Pa \cdot 1\frac{m^3}{s} = 1\frac{N}{m^2} \cdot \frac{m^3}{s}$$

$$1W = 1\frac{Nm}{s} = 1\frac{kg \cdot m^2}{s^3}$$

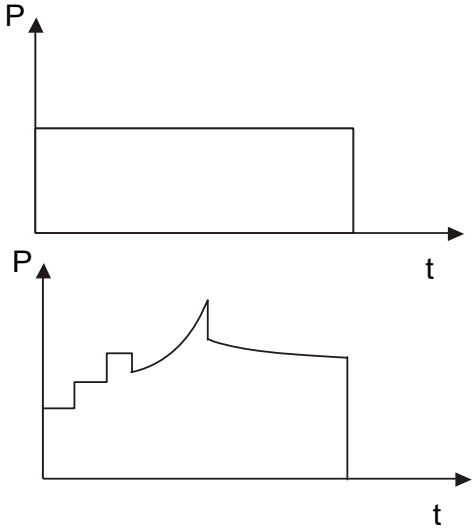
$$\underline{\underline{1Ws = 1Nm}}$$

Definition of the unit "volt"

electrical energy = mechanical work

$$\boxed{P = U \cdot I = R \cdot I^2 = \frac{U^2}{R}} \quad \Leftarrow \quad \text{Further formula possibilities}$$

The mechanical work is to a high degree replaceable by the electrical energy. So we can deem:



Energy used = work carried out:

with a temporary constant output: $W = P \cdot t$

generally: $\boxed{W = \int P dt}$

$\mathrm{Unit\,[W]} = \mathrm{Ws} = \mathrm{J\ (Joule)}$

In every case energy (work) is clearly described as the area under the power's graph.

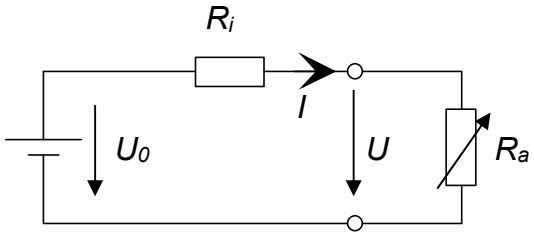
To explain the quantity of the SI-energy unit Joule, we compare it with a widely used unit in electrical power engineering:

$$1\,kWh = 10^3\,W \cdot 3600\,s = 3{,}6 \cdot 10^6\,Ws = 3{,}6 \cdot 10^6\,J$$

2.3.2 Power matching

We have a given technical voltage source, and we want to examine, under which condition we get the maximum power output from the source to the load R_a.

Technical voltage source:



given: U_0, R_i
looking for: P_a in dependency on R_a

Observation of the limit cases:

$$R_a = 0 \Rightarrow U = 0 \Rightarrow P_a = 0 \cdot I_K = 0$$

$$R_a = \infty \Rightarrow I = 0 \Rightarrow P_a = U_0 \cdot 0 = 0$$

The maximum of the power has to be between these two limit cases!

Approach: to write P_a as a function of R_a, then take the derivative and equate it to zero.

$$P_a = R_a \cdot I^2$$

$$I = \frac{U_0}{R_i + R_a}$$

$$P_a = \frac{R_a \cdot {U_0}^2}{(R_i + R_a)^2} = \frac{R_a \cdot {U_0}^2}{{R_i}^2 \cdot \left(1 + \frac{R_a}{R_i}\right)^2} = \frac{{U_0}^2}{R_i} \cdot \frac{\frac{R_a}{R_i}}{\left(1 + \frac{R_a}{R_i}\right)^2}$$

Definition:

$$y = \frac{x}{(1+x)^2} \quad \text{with} \quad x = \frac{R_a}{R_i} \quad \text{and} \quad y = \frac{P_a}{\frac{{U_0}^2}{R_i}}$$

Quotient's rule for derivatives:

$$y = \frac{u}{v} \qquad y' = \frac{u' \cdot v - u \cdot v'}{v^2}$$

Derivative:

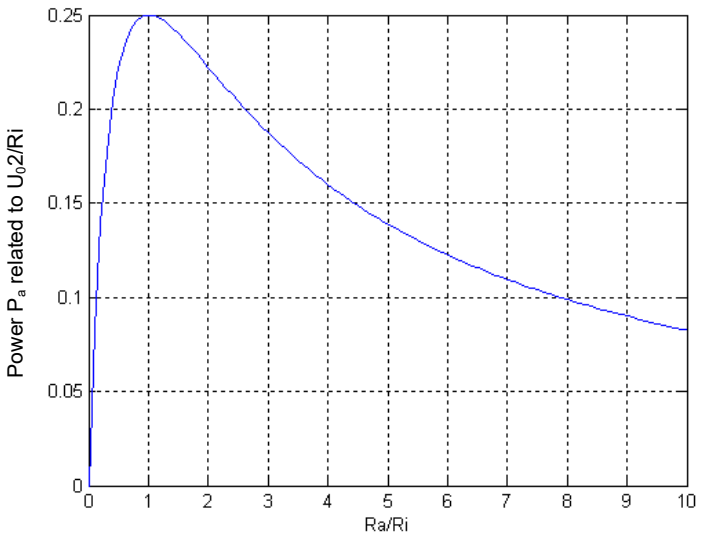
$$\frac{dy}{dx} = \frac{1 \cdot (1+x)^2 - x \cdot 2 \cdot (1+x)}{(1+x)^4} = 0 \Leftrightarrow (1+x)^2 = 2 \cdot x \cdot (1+x) \Leftrightarrow 1 + x = 2 \cdot x \Leftrightarrow$$

$$\underline{\underline{x = 1}} \quad \Rightarrow \quad \frac{R_a}{R_i} = 1 \quad \Rightarrow \quad \boxed{R_a = R_i} \quad \textbf{Maximum}$$

for $R_a = R_i$: $y_{max} = \frac{1}{(1+1)^2}$ $\underline{\underline{y_{max} = \frac{1}{4}}}$

Table of values:

R_a/R_i	$P_a/ U_0^2/R_i$	η/ %
0,1	0,083	9,1
0,2	0,139	16,7
0,5	0,222	33,3
0,75	0,244	42,9
1,0	0,250	50
1,5	0,240	60
2	0,222	66,7
3	0,187	75
4	0,160	80
5	0,139	83,3
7	0,109	87,5
10	0,083	90,9



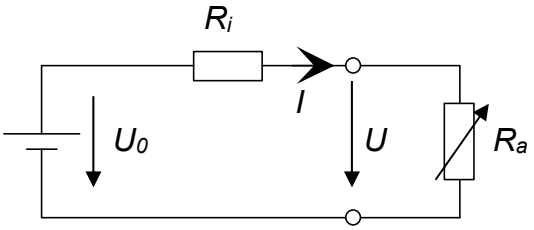
maximum at $R_a/R_i = 1$ $\Rightarrow$ $\boxed{R_a = R_i}$

maximum value: $P_{a,max}/U_0^2/R_i = 0{,}25$ $\Rightarrow$ $\boxed{P_{a,\max} = \frac{U_0^{\,2}}{4 \cdot R_i}}$

We finalize: The maximum power output from a given source is achieved for $R_a = R_i$. We call this condition power matching.

2.3.3 Efficiency

Another approach now is to investigate, how efficient a technical voltage source works when delivering the necessary power to a load. In other words: we want to get the relation between the effective power and the losses. For that we investigate exactly the same circuit like before (2.3.2).



$$\eta = \frac{\text{benefit (use)}}{\text{effort (consumption)}} = \frac{output}{input} \qquad \text{here:} \quad \eta = \frac{P_a}{P_i + P_a}$$

effective power: $P_a = R_a \cdot I^2 = \frac{R_a}{(R_i + R_a)^2} \cdot U_0^{\,2}$ $\qquad I = \frac{U_0}{R_i + R_a}$

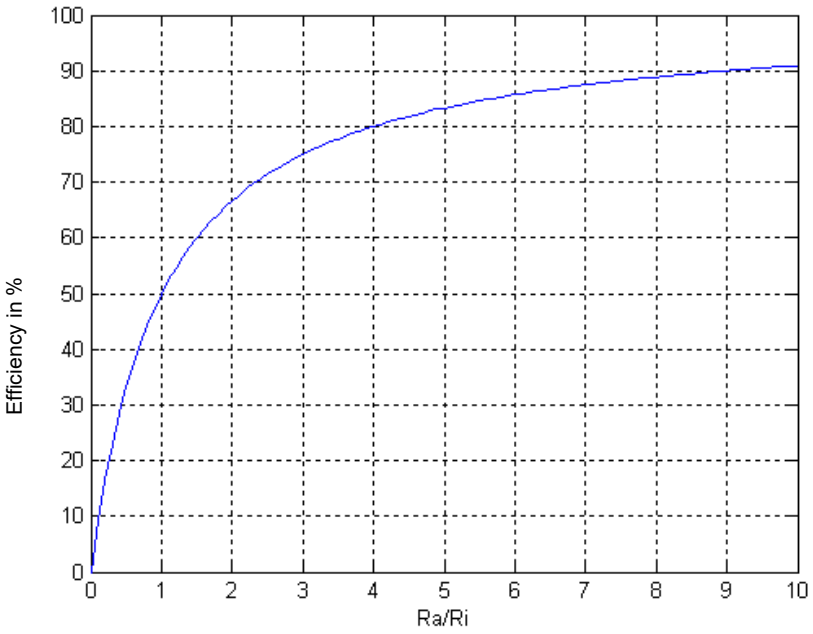
dissipation power (losses): $P_i = R_i \cdot I^2 = \frac{R_i}{(R_i + R_a)^2} \cdot U_0^{\,2}$

overall power: $P_i + P_a = (R_a + R_i) \cdot I^2 = \frac{R_a + R_i}{(R_i + R_a)^2} \cdot U_0^{\,2} = \frac{U_0^{\,2}}{R_i + R_a} = U_0 \cdot I$

$$\eta = \frac{\frac{R_a}{(R_i + R_a)^2} \cdot U_0^{\,2}}{\frac{R_a + R_i}{(R_i + R_a)^2} \cdot U_0^{\,2}}$$

$$\underline{\underline{\eta = \frac{R_a}{R_i + R_a}}}$$

$$\underline{\underline{\eta = \frac{\frac{R_a}{R_i}}{1 + \frac{R_a}{R_i}}}}$$



In case of matching we get:

$$R_a = R_i \Rightarrow \underline{\underline{\eta = 0{,}5}}$$

We again finalize: The higher the ratio R_a/R_i, the better is the efficiency. Obviously, there is a conflict now between the approaches of power matching and efficiency. In the technical reality we often have to find a compromise, which depends on the overall conditions. As a very simple approach we can distinguish:

Electrical power engineering: The generated electrical energy costs money and should be used as perfectly (efficiently) as possible.
Target: High efficiency $\Rightarrow$ $\eta \rightarrow 1$

Information technology: The voltage sources or supplies, resp. (electronic devices) cost money; the user wants to get as much power as possible from the voltage sources.
Target: Power matching $\Rightarrow$ $R_a = R_i$

2.3.4 Use of electrical heat

The so-called “Joule’s Law” (James Prescott Joule, 1841) gives the heat quantity Q, which is released by a current carrying electrical conductor:

$$Q \sim R \cdot I^2 \cdot t$$

However, this is not really a new law, because this quantity of heat is identical to the delivered thermal energy:

$$Q = W_{therm} = P \cdot t = R \cdot I^2 \cdot t \qquad \text{for } I \neq f(t)$$

$$Q = W_{therm} = \int p \cdot dt = R \cdot \int i^2 \cdot dt \qquad \text{in general}$$

Hence, the proportionality factor in Joule's Law is simply 1.

Remark: We use small letters ***u, i, p*** for voltage, current and power which vary with time.

Practical heating elements

Here, electrical heat is required. Due to the loss of radiation and the dissipation heat in the supply wires the efficiency is always $\eta < 100\%$.

$$\eta = \frac{P_{use}}{P_{total}}$$

$$P_{use} = \eta \cdot P_{total} = \eta \cdot P_{el}$$

$$W_{use} = \eta \cdot P_{el} \cdot t$$

for $P_{el} \neq f(t)$

Relationship between the supplied heat quantity and the increase in temperature:

$$W_{use} = m \cdot c \cdot \Delta T$$

c = specific heat of the material to be heated.

Remark 1: Sometimes an old unit still can be found, which should not be used any more:
1 kcal = 4186,8 J = 1,163*10^{-3} kWh

Remark 2: The quantities "work" and "energy" are interchangeable up to a certain extent (see 2.3.1):

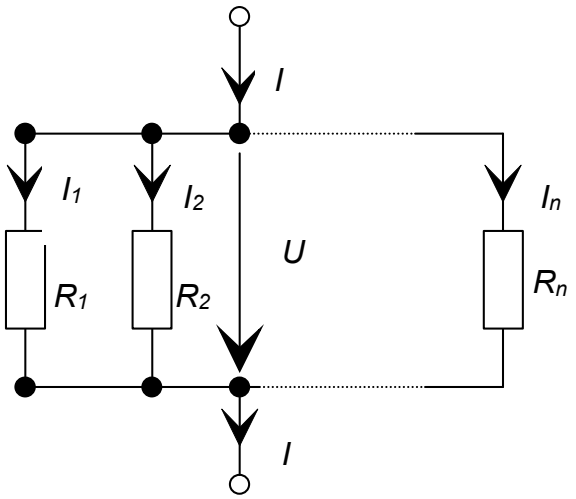
work = use of energy

energy content of a system = work available from the system

Example:
to pull a weight of 1N (more exact: a mass with a weight of 1N) up 1m you have to give a mechanical work of W_{mech} = 1Nm. Then the weight has the (additional) potential energy W_{pot} = 1J = 1Nm.

2.4 Calculation of linear networks

2.4.1 Parallel connection of resistances; Current divider



Qualitative understanding:
The current finds a further available path parallel to the smallest of the given resistances:
The total resistance is smaller than the smallest partial resistance.

$$I = I_1 + I_2 + ... + I_n$$

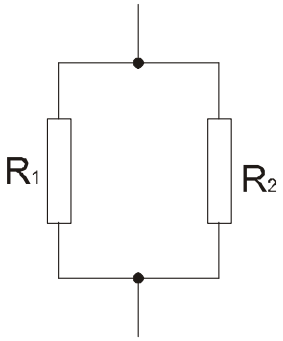
$$I = \frac{U}{R_1} + \frac{U}{R_2} + ... + \frac{U}{R_n}$$

$$G = \frac{1}{R} = \frac{I}{U} = \frac{1}{R_1} + \frac{1}{R_2} + ... + \frac{1}{R_n}$$

$$\boxed{\frac{1}{R} = \sum_{\nu=1}^{n} \frac{1}{R\nu}} \qquad \boxed{G = \sum_{\nu=1}^{n} G\nu}$$

In case of a parallel connection of resistances the electric conductivities are added

Useful formula for the parallel connection of two resistances:

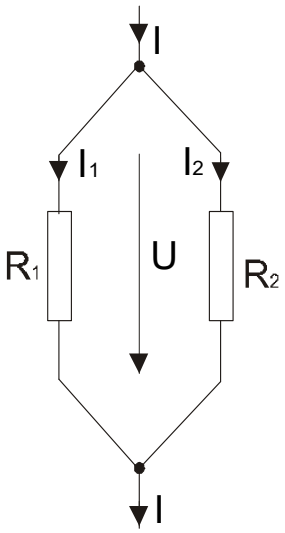


$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} = \frac{R_2 + R_1}{R_1 \cdot R_2}$$

$$R = \frac{R_1 \cdot R_2}{R_1 + R_2}$$

(be careful: this equation is valid only, if two resistances are connected in parallel)

Current divider: Parallel connection of resistances for the division (reduction) of an existing current *I*.



$$I = G \cdot U = (G_1 + G_2) \cdot U = \left(\frac{1}{R_1} + \frac{1}{R_2}\right) \cdot U$$

$$I_1 = G_1 \cdot U = \frac{1}{R_1} \cdot U$$

$$I_2 = G_2 \cdot U = \frac{1}{R_2} \cdot U$$

$$\frac{I_1}{I} = \frac{G_1}{G_1 + G_2} \qquad \frac{I_2}{I} = \frac{G_2}{G_1 + G_2} \qquad \frac{I_1}{I_2} = \frac{G_1}{G_2}$$

Partial current / Total current = Partial conductance / Total conductance
Partial current / Partial current = Partial conductance / Partial conductance

For the current divider the currents relate to each other like the corresponding conductances (of the current-carrying resistors)

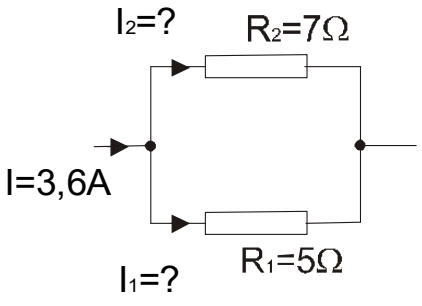
Formulation of the current dividing relationships using the resistance values:

$$\frac{I_1}{I} = \frac{\frac{1}{R_1}}{\frac{1}{R_1} + \frac{1}{R_2}} \qquad \frac{I_2}{I} = \frac{\frac{1}{R_2}}{\frac{1}{R_1} + \frac{1}{R_2}} \qquad \frac{I_1}{I_2} = \frac{\frac{1}{R_1}}{\frac{1}{R_2}} = \frac{R_2}{R_1}$$

(Expandable: Can also be used if there are more than two parallel resistances)

Current dividers are mostly used to solve measurements problems.

Example: Down-grading of currents

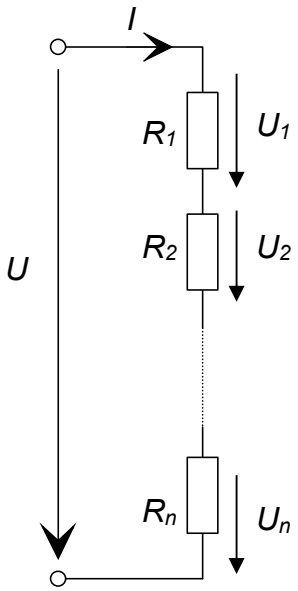


$$\frac{I_1}{I} = \frac{G_1}{G_1 + G_2} = \frac{\frac{1}{5\Omega}}{\frac{1}{5\Omega} + \frac{1}{7\Omega}} = \frac{7}{7+5} = \frac{7}{12}$$

$$I_1 = \frac{7}{12} \cdot I = \frac{7}{12} \cdot 3{,}6A = 2{,}1A = I_1$$

$$I_2 = I - I_1 = 3{,}6A - 2{,}1A = 1{,}5A = I_2$$

2.4.2 Series connections of resistances; Voltage divider



$$U = U_1 + U_2 + \ldots + U_n$$

$$U = R_1 \cdot I + R_2 \cdot I + \ldots + R_n \cdot I$$

$$\mathrm{R} = \frac{\mathrm{U}}{\mathrm{I}} = R_1 + R_2 + \ldots + R_n$$

$$\boxed{R = \sum_{\nu=1}^{n} R\nu}$$

For a series connection of resistances the resistance values are added.

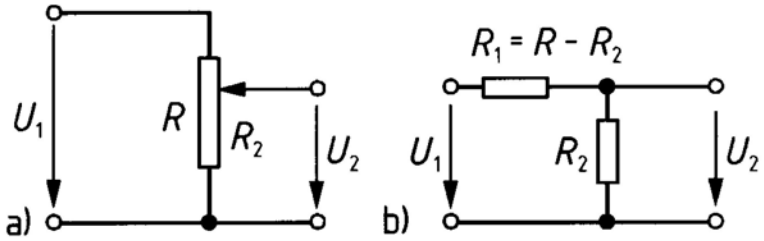
Voltage divider

$$\left.\begin{aligned} U &= R \cdot I \\ U_1 &= R_1 \cdot I \\ U_2 &= R_2 \cdot I \\ &\cdot \\ &\cdot \\ &\cdot \\ U_n &= R_n \cdot I \end{aligned}\right\} \qquad \frac{U_1}{U} = \frac{R_1}{R} \qquad \frac{U_2}{U} = \frac{R_2}{R} \qquad \frac{U_1}{U_2} = \frac{R_1}{R_2}$$

Partial voltage / Total voltage = Partial resistance / Total resistance
Partial voltage / Partial voltage = Partial resistance / Partial resistance

For the voltage divider the voltages are related similarily to the values of the resistances, at which they appear.

Practically realised voltage dividers are often variable (in steps or stepless, discrete or continuous):



Slide resistor or rotatable resistor (potentiometer).

Often definition:

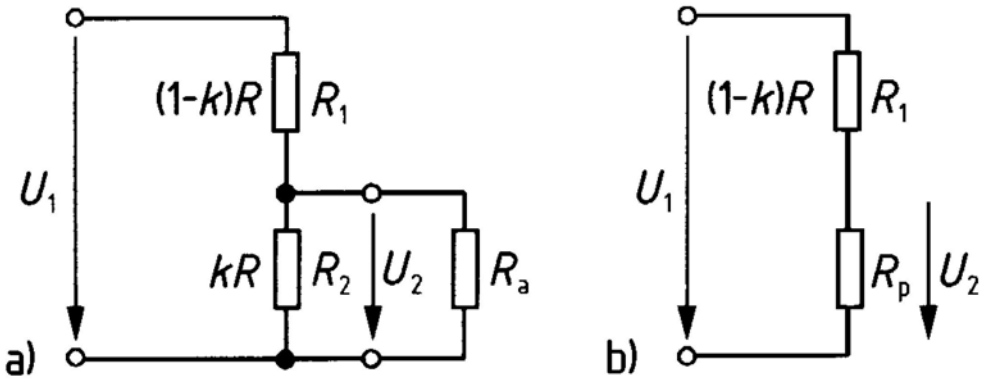
$$R_2 = k \cdot R$$

The value of *k* can be read of the scale.

$$U_2 = k \cdot U_1 \quad with \quad 0 \le k \le 1 \quad (no-load\ condition)$$

Voltage dividers are mostly used for measurement problems and also in the field of power supplies for electronic circuits. However, generally they have a poor efficiency.

Voltage divider with load R_a



$$\frac{U_2}{U_1} = \frac{R_p}{R_1 + R_p} = \frac{\frac{R_2 \cdot R_a}{R_2 + R_a}}{R_1 + \frac{R_2 \cdot R_a}{R_2 + R_a}} = \frac{R_2 \cdot R_a}{R_1 \cdot R_2 + R_1 \cdot R_a + R_2 \cdot R_a} = \frac{R_2}{\frac{R_1 \cdot R_2}{R_a} + R_1 + R_2}$$

$$for\ R_a \to \infty\ (no-load\ condition): \quad \frac{U_2}{U_1} = \frac{R_2}{R_1 + R_2}$$

Use of quantity *k*:

$$k = \frac{R_2}{R_1 + R_2} = \frac{R_2}{R} \qquad \Rightarrow \qquad \begin{aligned} R_2 &= k \cdot R \\ R_1 &= (1-k) \cdot R \end{aligned}$$

$$\frac{U_2}{U_1} = \frac{k \cdot R}{\frac{(1-k) \cdot R \cdot k \cdot R}{R_a} + R}$$

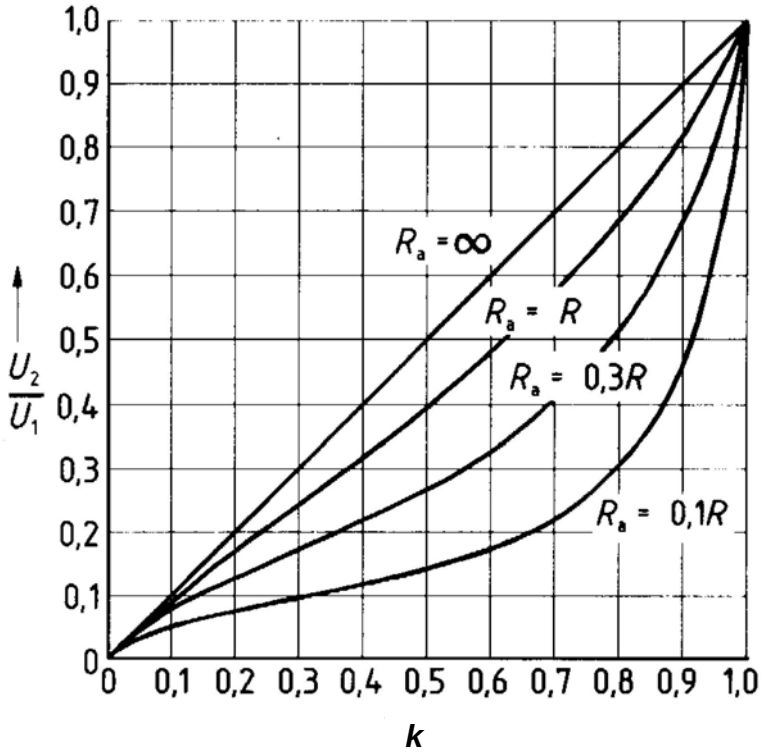
$$\frac{U_2}{U_1} = \frac{k}{\frac{(1-k) \cdot k \cdot R}{R_a} + 1}$$

$$\frac{U_2}{U_1} = \frac{k}{1 + (1-k) \cdot k \cdot \frac{R}{R_a}}$$

$for\ R_a \to \infty\ (no-load):$

$$\frac{U_2}{U_1} = k$$

For the voltage divider with load a nonlinear relationship exists between the resistance ratio (quantity *k*) and the voltage ratio U_2/U_1 (see figure).



2.4.3 Kirchhoff's Laws

The following presents the fundamental laws of circuit analysis, which serve as the foundation for electrical circuits' study and for all the methods to analyse these circuits. The basic laws of the electrical circuit analysis are called Kirchhoff's Laws (Gustav Robert Kirchhoff, 1824 -. 1887), dealing with currents and voltages in the circuit.

As a preparation for this analysis we must first give some definitions of the electrical circuit.

Remark: The investigations and the presented rules and equations are valid also in non-linear networks (electrical circuits which contain also non-linear elements).

Definitions:

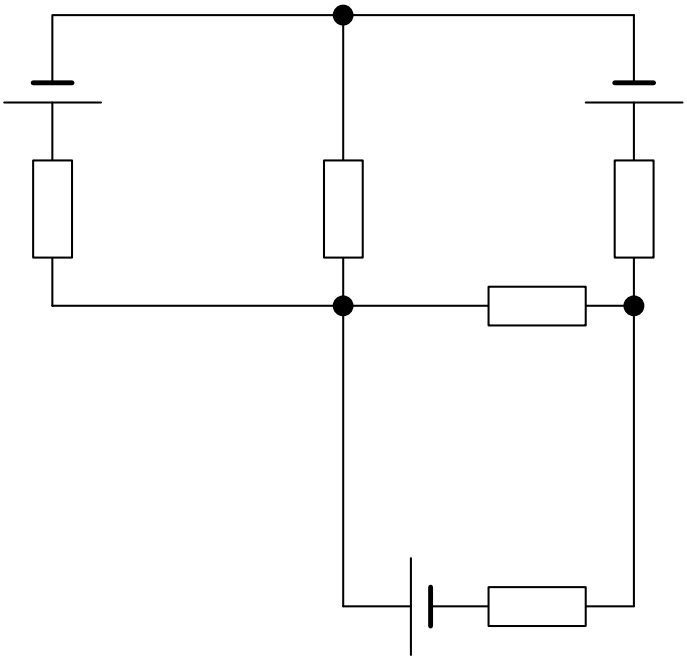
Network Interconnection of active and/or passive two-terminals

Node Point of current distribution

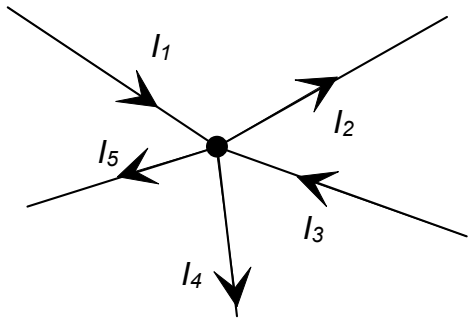
Branch direct connection between two nodes

Mesh simply closed way within the network along existing branches

Example of a simple network:



Node rule:



$\sum I_{in} = \sum I_{out}$

Here for this example:

$I_1 + I_3 = I_2 + I_4 + I_5$

Common formulation:

$$\sum I_\nu = 0$$

Kirchhoff's current law (KCL)

At a node the sum of all the currents is zero

Condition: With reference to the node all the current directional arrows are related to the same direction. Currents flowing in opposite direction are set with a negative polarity sign.

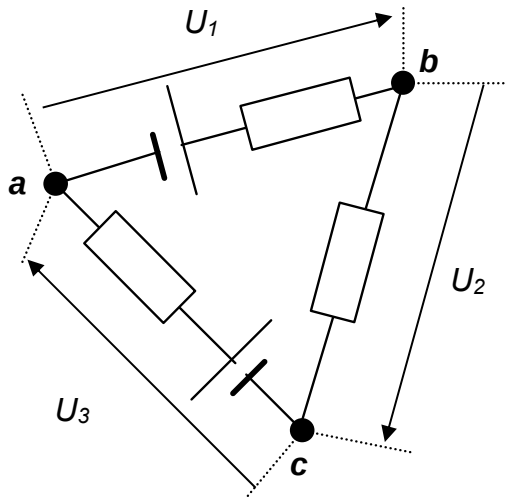
For the example above there has to be defined:

- either: in direction to the node as positive $\Rightarrow$ $I_1 - I_2 + I_3 - I_4 - I_5 = 0$

- or: in direction from the node as positive $\Rightarrow$ $-I_1 + I_2 - I_3 + I_4 + I_5 = 0$

Both equations lead to the same result !

Mesh rule:



$U_1 = U_{ab} = \varphi_a - \varphi_b$

$U_2 = U_{bc} = \varphi_b - \varphi_c$

$U_3 = U_{ca} = \varphi_c - \varphi_a$

$U_1 + U_2 + U_3 = 0$

U_{xy}: Voltage between two points

φ_x: Potential of the point (voltage between the point and earth)

Common formulation:

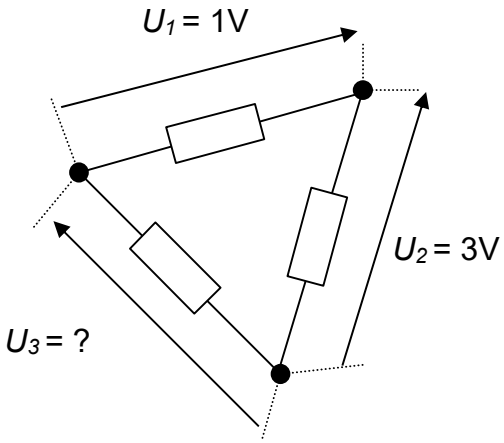
$$\sum U_\nu = 0$$

Kirchhoff's voltage law (KVL)

In a mesh the total sum of all voltages is zero

Condition: All the voltage directional arrows „follow each other", i.e. each tip of an arrow has to be connected to the end of the next arrow for a positive polarity sign. Voltages with different arrow directions are set with a negative polarity sign.

Example:



either: positive directional arrows
→ clockwise direction

$U_1 - U_2 + U_3 = 0$

or: positive directional arrows
→ anti-clockwise direction

$U_2 - U_1 - U_3 = 0$

Both equations lead to the same result:

$U_3 = U_2 - U_1$

$U_3 = 3\ V - 1\ V$

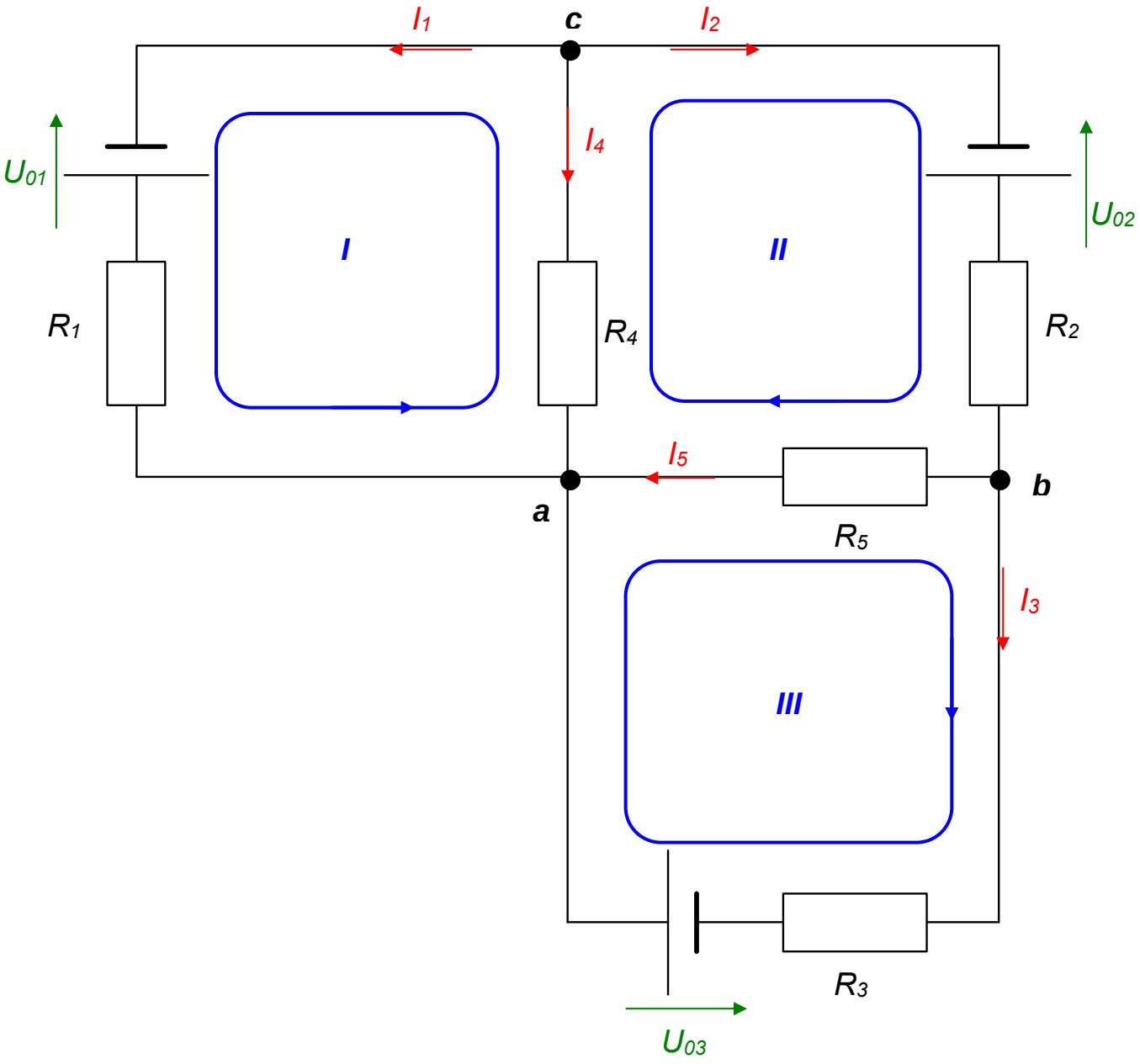
$\underline{U_3 = 2\ V}$

Another formulation of this result:

The voltage between two points is independent from the pathway

2.4.4 Node and mesh analysis

We discuss the general procedure to analyse an electrical circuit. As an example we consider the following network, for which the three open-circuit voltages U_{01} … U_{03} and the five resistances R_1 … R_5 are given:



The network has: $k = 3$ nodes
$z = 5$ branches
$\Rightarrow$ 5 unknown currents

Application of node and mesh rules:

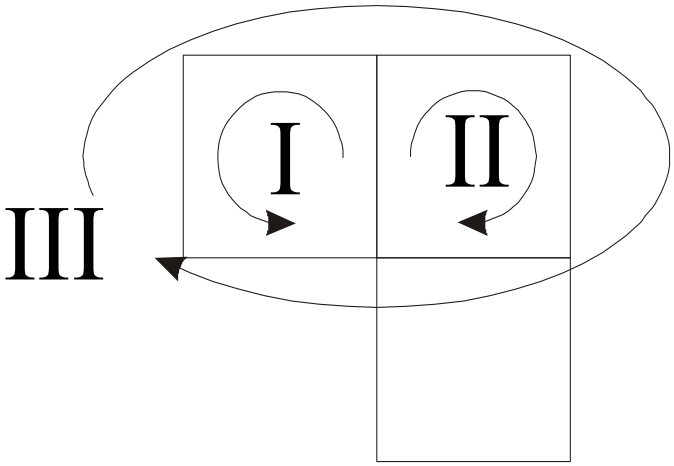
Node a	I_1		$+ I_3$	$+ I_4$	$+ I_5$	= 0
Node b		I_2	$- I_3$		$- I_5$	= 0
Mesh I	$R_1 \cdot I_1$			$- R_4 \cdot I_4$	$- U_{01}$	= 0
Mesh II		$R_2 \cdot I_2$		$- R_4 \cdot I_4$	$+ R_5 \cdot I_5 - U_{02}$	= 0
Mesh III			$R_3 \cdot I_3$		$- R_5 \cdot I_5 - U_{03}$	= 0

Remarks:

Node a/b: Current direction to node $\rightarrow$ positive
(it is, of course, also possible, to define the opposite current direction as positive)

Node c: $I_1 + I_2 + I_4 = 0$ (= a redundant equation)
This follows directly from the addition of both the given node equations: no new information is obtained. (**Linear dependency** of the equations). Generally a new current appears in each new node equation, only in the last one this does not happen. Therefore only ***k*-1 node equations** are useful.

z branches: The rest of the equations necessary for the calculation of the *z* unknown currents are: *z* - (*k*-1) = ***z* - *k* + 1 mesh equations**. Here for example *z* - *k* + 1 = 5 - 3 + 1 = 3 mesh equations are necessary.
The meshes have to be selected in such a way, that the mesh equations are **not linearly dependent** on each other. Linear dependency of the mesh equations appears for example with a selection of meshes, as indicated here:
For the selection of meshes being linearly independent see the concept of the **complete tree**.



General rules for the application of the node and mesh analysis:

- *k* and *z* are determined
- current directional arrows are fixed (arbitrarily)
- *k* - 1 node equations are formulated
- *z* - *k* + 1 meshes are selected
- positive voltage directional arrows are fixed
- *z* - *k* + 1 mesh equations are formulated, (linearly independent)
- system of equations is arranged
- system of equations is solved

Mesh selection - the complete tree:

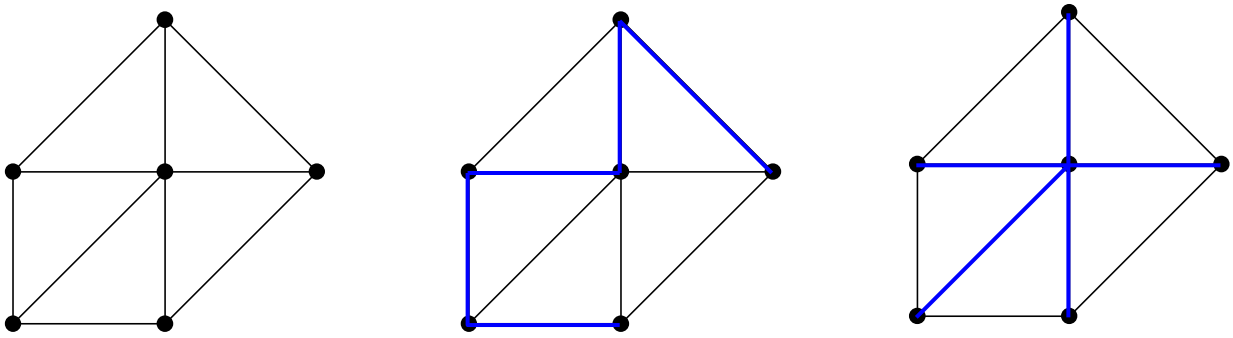
For the selection of z - k + 1 meshes, which lead to mesh equations, which are linearly independent on each other.

> **A complete tree is one open (not-closed) connection of all nodes along the existing branches.**

Example:

Graph of a network

Two possibilities for a complete tree
(from about 120 for this example)



A complete tree contains k - 1 branches.

The remaining z - k + 1 branches which do not belong to the complete tree are called the connecting branches.

Selection of meshes: Every selected mesh contains exactly one connecting branch, all other branches belong to the complete tree.

Meshes chosen in this way always deliver mesh equations, which are linearly independent on each other. After selecting the complete tree the meshes to be used are exactly fixed.

If particular branches shall become connecting branches (e.g. for the method of mesh currents, because then the currents flowing in the connecting branches become the mesh currents), these branches have to be omitted by the assembly of the complete tree.

2.4.5 Superposition

For a **linear network** the superposition principle is valid.
(A linear network contains only linear elements in the equivalent circuit, additionally to the ideal sources. Linear element means: U is proportional to I)

Superposition principle

If a linear network contains more than one source, the partial currents originating from the individual sources can be calculated separately. The total current in any path of the network is the addition (superposition) of all existing partial currents, taking into account the current direction (correct sign).

Procedure

The current at the location of interest in the network can be calculated:

1 only the 1st voltage (or current) source is taken into account, all other ideal voltage sources are replaced by short circuits, all other ideal current sources by open circuits

2 only the 2nd voltage (or current) source is taken into account, all other ideal voltage sources are replaced by short circuits, all other ideal current sources by open circuits

…

n only the last (n) voltage (or current) source is taken into account, all other ideal voltage sources are replaced by short circuits, all other ideal current sources by open circuits

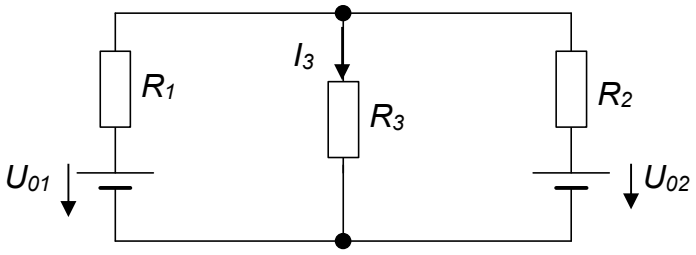
Finally: all n partial currents have to be added together.

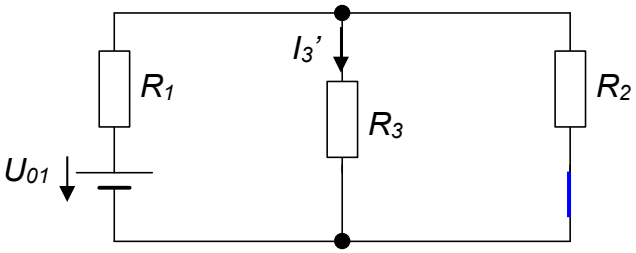
Remark:
If sources or their influence in a circuit have to be neglected (not be taken into account), the procedure is always the same (not only here for the superposition principle):

- ideal voltage sources have to be replaced by short circuits: the short circuit is the only possible element, which always and under all conditions shows a voltage $U = 0$, which is necessary here to neglect any influence of the ideal voltage source) – an ideal voltage source is a technical voltage source with $R_i = 0 \rightarrow$ short circuit

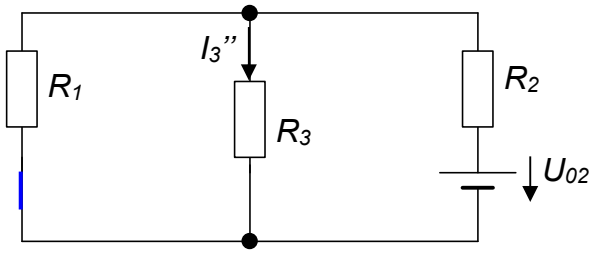
- ideal current sources have to be replaced by open circuits: the open circuit is the only possible element, which always and under all conditions shows a current $I = 0$, which is necessary here to neglect any influence of the ideal current source) - an ideal current source is a technical current source with $R_i = \infty \rightarrow$ open circuit

Example: We assume, we are interested in the current I_3





source 1 active,
source 2 replaced by a short circuit



source 2 active,
source 1 replaced by a short circuit

$$I_3' = \frac{\frac{1}{R_3}}{\frac{1}{R_2}+\frac{1}{R_3}} \cdot \frac{U_{01}}{R_1 + \frac{1}{\frac{1}{R_2}+\frac{1}{R_3}}}$$

$$I_3'' = \frac{\frac{1}{R_3}}{\frac{1}{R_1}+\frac{1}{R_3}} \cdot \frac{U_{02}}{R_2 + \frac{1}{\frac{1}{R_1}+\frac{1}{R_3}}}$$

$$I_3' = \frac{\frac{1}{R_3} \cdot U_{01}}{\frac{R_1}{R_2}+\frac{R_1}{R_3}+1} = \frac{R_2 \cdot U_{01}}{R_1 \cdot R_3 + R_1 \cdot R_2 + R_2 \cdot R_3}$$

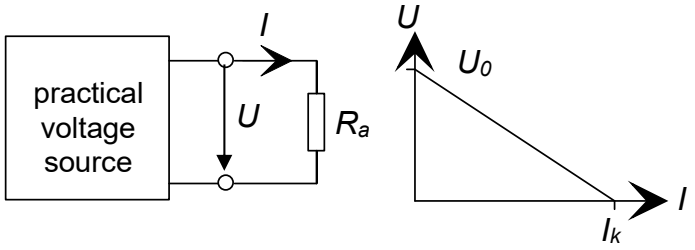
$$I_3'' = \frac{R_1 \cdot U_{02}}{R_2 \cdot R_3 + R_2 \cdot R_1 + R_1 \cdot R_3}$$

$$I_3 = I_3' + I_3'' = \frac{R_2 \cdot U_{01} + R_1 \cdot U_{02}}{R_1 \cdot R_2 + R_2 \cdot R_3 + R_1 \cdot R_3}$$

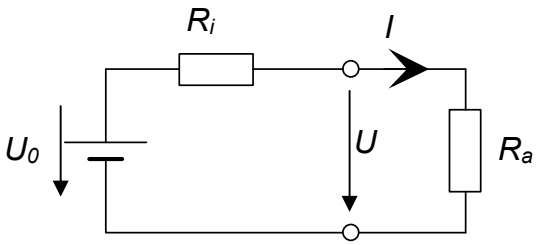
2.4.6 Equivalent voltage or current source

We know, that a technical voltage (or current) source at the terminals shows a linear behaviour of voltage and current (see 2.2.3 and 2.2.4). Connected to the terminals in this case is only one resistance R_a.

Technical voltage source with linear *U-I* behaviour



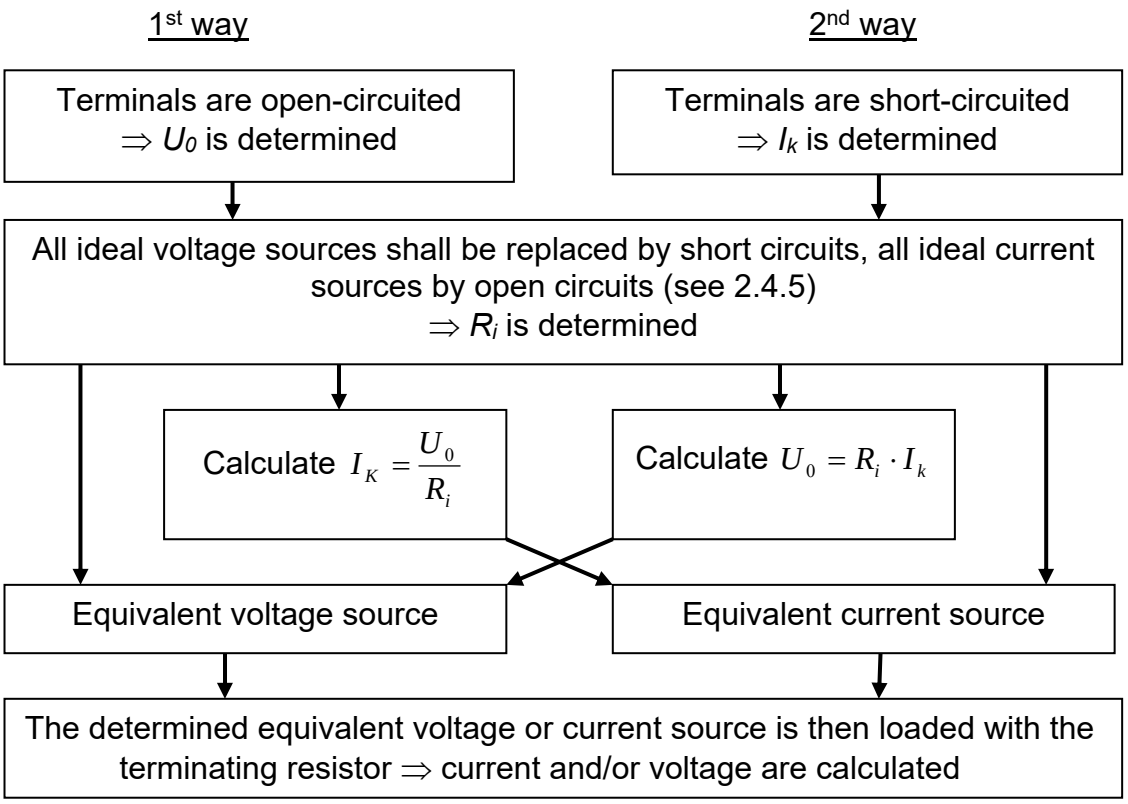
Equivalent voltage source



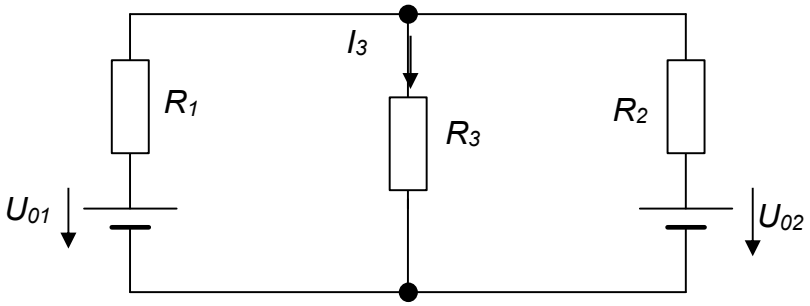
The internal circuits of the practical voltage source are unknown; this might be also a very complicated linear network.

Procedure: We assume, we want to calculate current and/or voltage only for one certain resistance in the given network. This resistance is taken as the terminating resistance. The entire rest of the network is interpreted as a technical voltage source, for which the equivalent circuit has to be found.

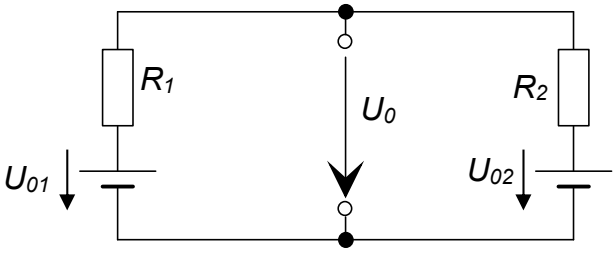
Determination of the equivalent voltage (or current) source elements:



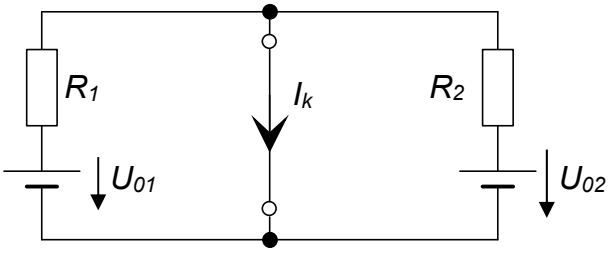
Example: I_3 = ?



1st way



2nd way



U_0 $(R_1 + R_2) \cdot I' + U_{02} - U_{01} = 0$

$$I' = \frac{U_{01} - U_{02}}{R_1 + R_2}$$

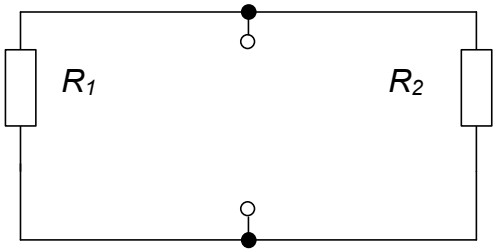
$$U_0 = R_2 \cdot I' + U_{02}$$

$$U_0 = \frac{R_2 \cdot U_{01} - R_2 \cdot U_{02} + (R_1 + R_2) \cdot U_{02}}{R_1 + R_2}$$

$$U_0 = \frac{R_2 \cdot U_{01} + R_1 \cdot U_{02}}{R_1 + R_2}$$

I_k $I_k = \frac{U_{01}}{R_1} + \frac{U_{02}}{R_2}$

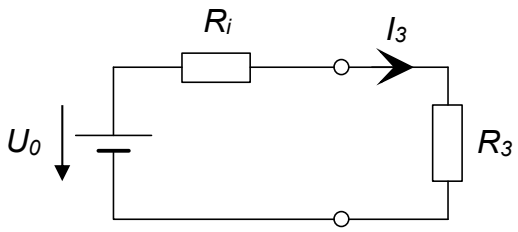
R_i $R_i = \frac{1}{\frac{1}{R_1} + \frac{1}{R_2}} = \frac{R_1 \cdot R_2}{R_1 + R_2} = R_i$



$$U_0 = R_i \cdot I_K$$

$$U_0 = \frac{R_2 \cdot U_{01} + R_1 \cdot U_{02}}{R_1 + R_2}$$

Finally we get an equivalent voltage source loaded with the terminating resistance R_3:



$$I_3 = \frac{U_0}{R_i + R_3} = \frac{R_2 \cdot U_{01} + R_1 \cdot U_{02}}{(R_1 + R_2) \cdot \left(\frac{R_1 \cdot R_2}{R_1 + R_2} + R_3 \right)}$$

$$I_3 = \frac{R_2 \cdot U_{01} + R_1 \cdot U_{02}}{R_1 \cdot R_2 + R_1 \cdot R_3 + R_2 \cdot R_3}$$

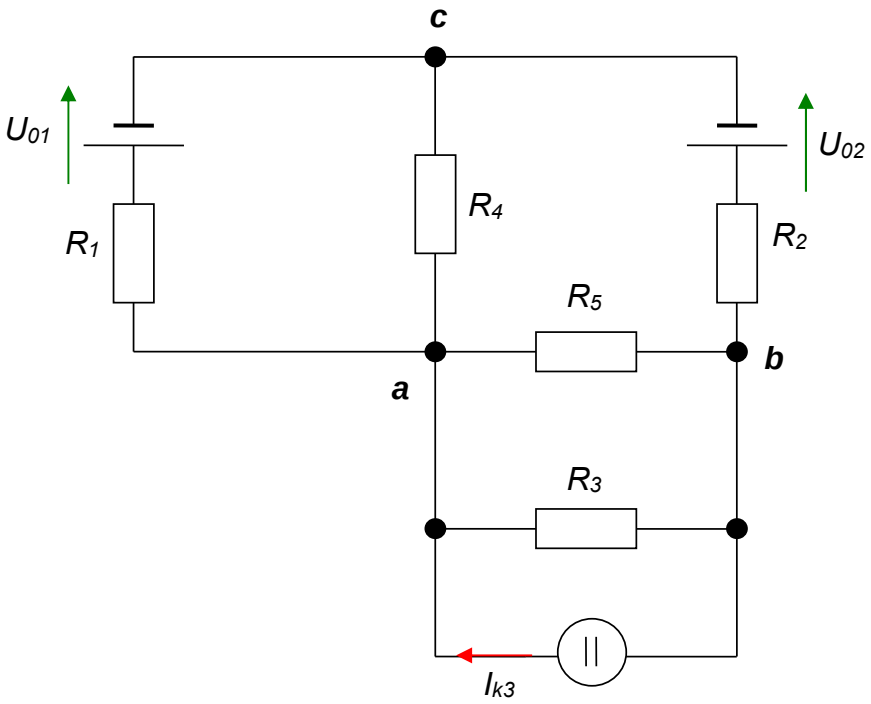
2.4.7 Mesh current analysis

For this method, only the (*z - k + 1)* mesh equations are required, no more node equations. The information coming from the node equations is automatically taken into account by the definition of the mesh currents.

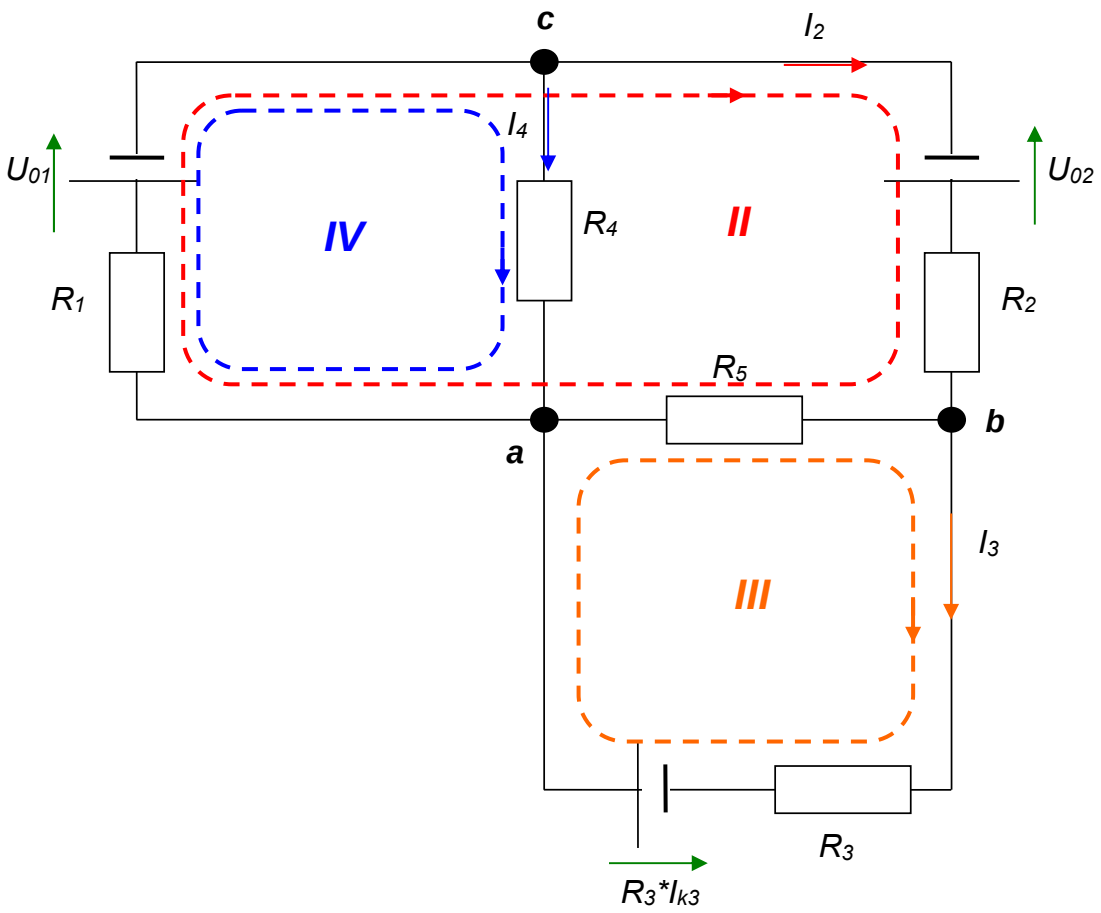
Firstly all the current sources have to be converted to equivalent voltage sources. The currents in the connecting branches (see method of the complete tree) are called **mesh currents**.

Example:

Original circuit



Circuit after conversion (contains only voltage sources):



Method of calculation:

A complete tree is defined connecting all nodes of the network. The branches not belonging to the complete tree are the connecting branches.

Each mesh current always flows through the entire mesh. In the connecting branches only the mesh currents concerned flow.

The other branches contain more than only one mesh current. These branches are parts of the complete tree. The actual currents of these branches are given by the superposition of the mesh currents $\rightarrow \; I_1 = -I_2 - I_4\,, \quad I_5 = I_2 - I_3$

Mesh II $\qquad R_2 \cdot I_2 + R_5 \cdot (I_2 - I_3) + R_1 \cdot (I_2 + I_4) + U_{01} - U_{02} = 0$

Mesh III $\qquad R_5 \cdot (I_3 - I_2) + R_3 \cdot I_3 - R_3 \cdot I_{k3} = 0$

Mesh IV $\qquad R_4 \cdot I_4 + R_1 \cdot (I_4 + I_2) + U_{01} = 0$

System of equations:

$$\begin{bmatrix} (R_1+R_2+R_5) & -R_5 & R_1 \\ -R_5 & (R_3+R_5) & 0 \\ R_1 & 0 & (R_1+R_4) \end{bmatrix} \cdot \begin{bmatrix} I_2 \\ I_3 \\ I_4 \end{bmatrix} = \begin{bmatrix} U_{02}-U_{01} \\ R_3 \cdot I_{k3} \\ -U_{01} \end{bmatrix}$$

The resistance matrix is always symmetric to the main diagonal, if the orientation of the mesh cycle (II, III, IV) and the mesh current directional arrow are codirectional (have the same direction).

Main diagonal: **Sum of the resistances**
(all resistances in the mesh concerned)

Remaining places: **Coupling resistances**
(resistances where the mesh currents I_μ and I_ν flow simultaneously)
- positive, if the mesh currents flow through the resistance codirectionally
- negative, if the mesh currents flow through the resistance reversely

Result vector: **Sum of the voltage sources** in the mesh concerned
(with the original sign, if the directional voltage arrow shows reversely to the orientation of the mesh cycle, otherwise opposite sign)

2.4.8 Node voltage analysis

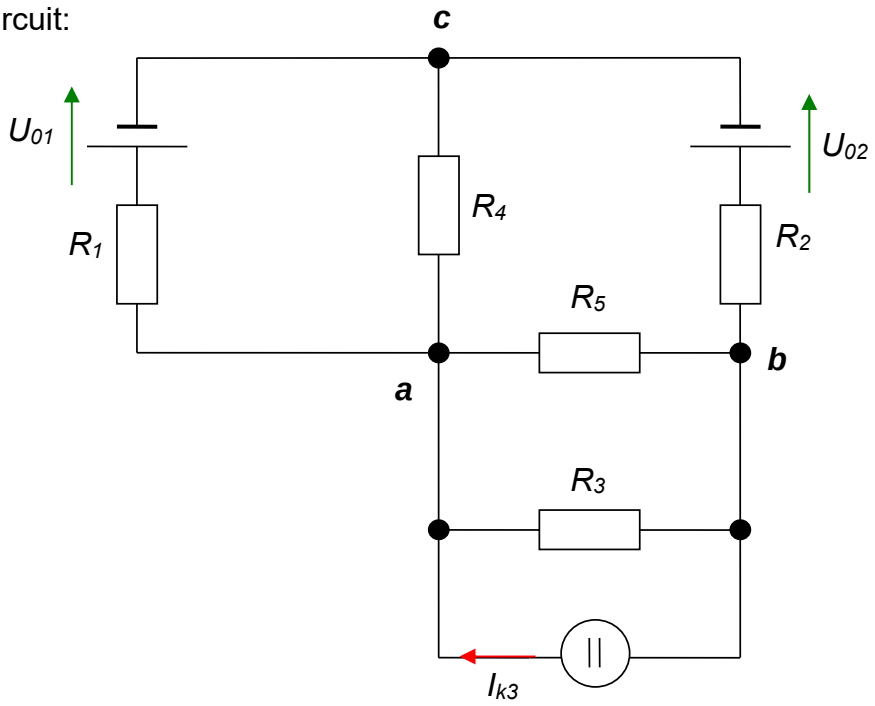
For this method only the (k – 1) node equations required, no more mesh equations. The information coming from the mesh equations is automatically taken into account by the definition of the node voltages. With that, this method is exactly opposite to the mesh current analysis (2.4.7).

Firstly all the voltage sources have to be converted to equivalent current sources. The potential of one node is chosen as the **reference potential**.

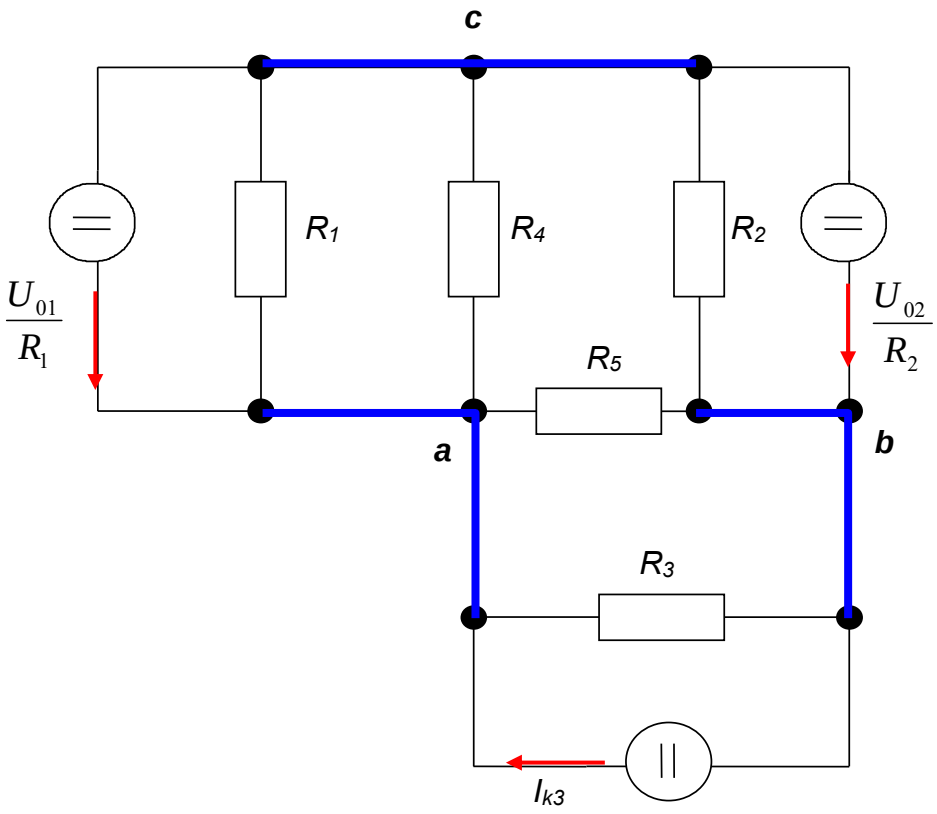
Useful: - no node equation for the reference potential,
- at all nodes the direction of the departing currents I_{out} is defined as positive (although this is not a necessity).

Example:

Original circuit:



Circuit after conversion (contains only current sources):



Method of calculation:

For all nodes (except the node building the reference potential) the node equations have to be arranged.

Here: Node **a** is chosen as the reference potential.

Node **b**
$$\left(\frac{1}{R_5}+\frac{1}{R_3}\right)\cdot U_B+\frac{1}{R_2}\left(U_B-U_C\right)-\frac{U_{02}}{R_2}+I_{k3}=0$$

Mode **c**
$$\frac{U_{01}}{R_1}+\left(\frac{1}{R_1}+\frac{1}{R_4}\right)\cdot U_C+\frac{1}{R_2}\left(U_C-U_B\right)+\frac{U_{02}}{R_2}=0$$

System of equations:

$$\begin{bmatrix}\left(\frac{1}{R_2}+\frac{1}{R_5}+\frac{1}{R_3}\right) & -\frac{1}{R_2}\\ -\frac{1}{R_2} & \left(\frac{1}{R_1}+\frac{1}{R_4}+\frac{1}{R_2}\right)\end{bmatrix}\cdot\begin{bmatrix}U_B\\U_C\end{bmatrix}=\begin{bmatrix}\frac{U_{02}}{R_2}-I_{k3}\\ -\frac{U_{01}}{R_1}-\frac{U_{02}}{R_2}\end{bmatrix}$$

The admittance matrix (consisting of the electric conductances) is always symmetric to the main diagonal, if there is no equation made for the potential reference point and the direction of the departing currents I_{out} is always calculated as positive.

Main diagonal: **Sum of the electric conductances**
(all conductances, which border to the node concerned)

Remaining places: **Coupling electric conductances** with negative sign
(all conductances, which are connected between the μ and ν nodes)

Result vector: **Sum of the current sources**, which are impressed at the node concerned (with the original sign, if the current directional arrow shows to the node, otherwise opposite sign).

2.4.9 Star - delta transformation

Star and delta (triangle) are the principle circuits, which are able to be built between three external terminals.



Aim: The circuits have to be dimensioned in such a way that their behaviour to the external terminals is equal $\rightarrow$ **Equivalence**

The resistances between two corresponding terminals have to be the same in each case:

Resistance between the terminals	**star**	**delta**	
1 and 2	$R_1 + R_2$	$= \frac{R_{12} \cdot (R_{23} + R_{31})}{R_{12} + R_{23} + R_{31}}$	(Eq. 1)
2 and 3	$R_2 + R_3$	$= \frac{R_{23} \cdot (R_{31} + R_{12})}{R_{23} + R_{31} + R_{12}}$	(Eq. 2)
3 and 1	$R_3 + R_1$	$= \frac{R_{31} \cdot (R_{12} + R_{23})}{R_{31} + R_{12} + R_{23}}$	(Eq. 3)

Remark: Every equation results from the proceeding equation (the first one also from the last) through the cyclic exchange of the indices:
$1 \rightarrow 2 \qquad 2 \rightarrow 3 \qquad 3 \rightarrow 1$

Now we build (Eq. 1) - (Eq. 2) + (Eq. 3):

$$2 \cdot R_1 = \frac{R_{12} \cdot R_{23} + R_{12} \cdot R_{31} - R_{23} \cdot R_{31} - R_{23} \cdot R_{12} + R_{31} \cdot R_{12} + R_{31} \cdot R_{23}}{R_{12} + R_{23} + R_{31}}$$

Conversion of DELTA to STAR:

$$R_1 = \frac{R_{12} \cdot R_{31}}{R_{12} + R_{23} + R_{31}}$$

$$R_2 = \frac{R_{23} \cdot R_{12}}{R_{12} + R_{23} + R_{31}}$$

$$R_3 = \frac{R_{31} \cdot R_{23}}{R_{12} + R_{23} + R_{31}}$$

R_1 in star-circuit and *R_{31}, R_{12}* in delta-circuit are connected to terminal 1

R_2 in star-circuit and *R_{12}, R_{23}* in delta-circuit are connected to terminal 2

R_3 in star-circuit and *R_{23}, R_{31}* in delta-circuit are connected to terminal 3

The equations for *R_2* and *R_3* can be derived from the equation for *R_1* again through cyclic exchange of the subscripts. From this the same denominator always follows.

The formulas for the transformation in the reversed direction follow from the formulas given above by solving them for *R_{12}, R_{23}, R_{31}*:

$$\frac{R_{12} + R_{23} + R_{31}}{R_{12} + R_{23} + R_{31}} \cdot R_{12} = R_{12}$$

$$\frac{R_{31}}{R_{31}} \cdot \frac{R_{12} \cdot R_{12}}{R_{12} + R_{23} + R_{31}} + \frac{R_{23} \cdot R_{12}}{R_{12} + R_{23} + R_{31}} + \frac{R_{31} \cdot R_{12}}{R_{12} + R_{23} + R_{31}} = R_{12}$$

$$\frac{R_{12}}{R_{31}} \cdot \frac{R_{31} \cdot R_{12}}{R_{12} + R_{23} + R_{31}} + \frac{R_{23} \cdot R_{12}}{R_{12} + R_{23} + R_{31}} + \frac{R_{31} \cdot R_{12}}{R_{12} + R_{23} + R_{31}} = R_{12}$$

$$\frac{R_2}{R_3} \cdot R_1 \quad + \quad R_2 \quad + \quad R_1 \quad = R_{12}$$

Conversion of STAR to DELTA:

$$R_{12} = R_1 + R_2 + \frac{R_1 \cdot R_2}{R_3}$$

$$R_{23} = R_2 + R_3 + \frac{R_2 \cdot R_3}{R_1}$$

$$R_{31} = R_3 + R_1 + \frac{R_3 \cdot R_1}{R_2}$$

R_{12} in delta-circuit and *R_1, R_2* in star-circuit are connected between the terminals 1 and 2

R_{23} in delta-circuit and *R_2, R_3* in star-circuit are connected between the terminals 2 and 3

R_{31} in delta-circuit and *R_3, R_1* in star-circuit are connected between the terminals 3 and 1

The equations for *R_{23}* and *R_{31}* can be derived from the equation for *R_{12}* again through cyclic exchange of the subscripts.

2.4.10 Comparison of the different network calculation methods

The node and mesh analysis (chapter 2.4.4) is the most basic calculation method. With this method in principle every network can be solved, but for larger networks the number of node and mesh equations increases, i.e. the calculation work may be too heavy.

We can always benefit from simplifying the network as far as possible before starting calculations. If the resistances cannot be combined simply by using the laws for parallel and/or series connections (chapter 2.4.1 and 2.4.2), sometimes a star-delta transformation can be useful (chapter 2.4.9).

The superposition method (chapter 2.4.5) is applicable for linear networks only. It is convenient especially for networks with a few sources, and if the influence of the individual sources can be calculated easily. Furthermore the superposition principle is important in many fields of electrical engineering.

Mesh current analysis (chapter 2.4.7) and node voltage analysis (chapter 2.4.8) allow network calculations with a smaller number of equations. However, usually some sources have to be converted first: the mesh current analysis is possible for a network which contains only voltage sources, the node voltage analysis is possible for a network which contains only current sources. The following questions may be helpful for making a clever choice: Does the original circuit contain more voltage or more current sources? Are voltages or currents required as the result?

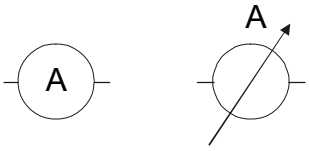
The method using an equivalent voltage or current source (chapter 2.4.6) is useful, if only current and/or voltage of a certain branch is required (the currents and voltages of other network’s branches then shall be not of interest).

2.5 Measurement of current, voltage, resistance and power

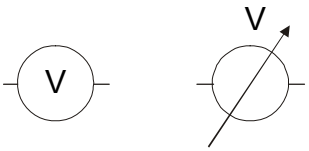
In this chapter only some general information and some basic circuits are presented due to the fact, that measurement technology is a major topic in all study courses of electrical engineering. Further information about measurement procedures, background and circuits is given in following study modules.

2.5.1 Principle circuits

Current meter (ammeter)



Voltmeter



The load, which is created by the measuring instrument itself, should not change remarkably the behaviours of the electrical circuit. Therefore we can formulate the following requirement:

$R_A << R_i + R_a$ $\qquad$ $R_V >> R_a$

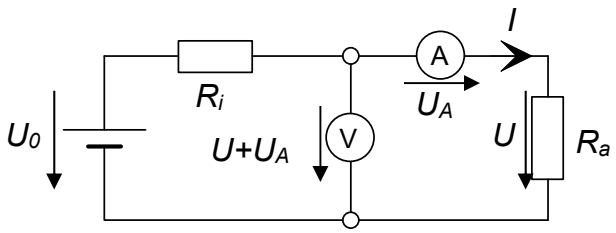
Typical values (for example):

R_A = 50 … 500 mΩ
in 10 … 1 A - measuring range

R_V = 10 MΩ
in 100 V - measuring range
(digital instruments)

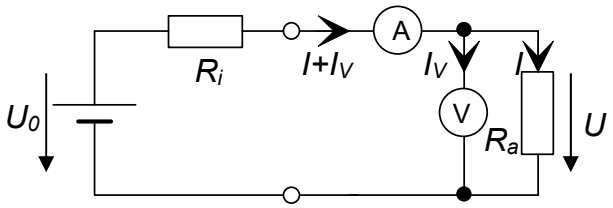
We have two general possibilities of simultaneously current and voltage measurements in an electrical circuit:

Current related measurement
for high ohmic load



$U_A << U$ if $R_a >> R_A$

Voltage related measurement
for low ohmic load

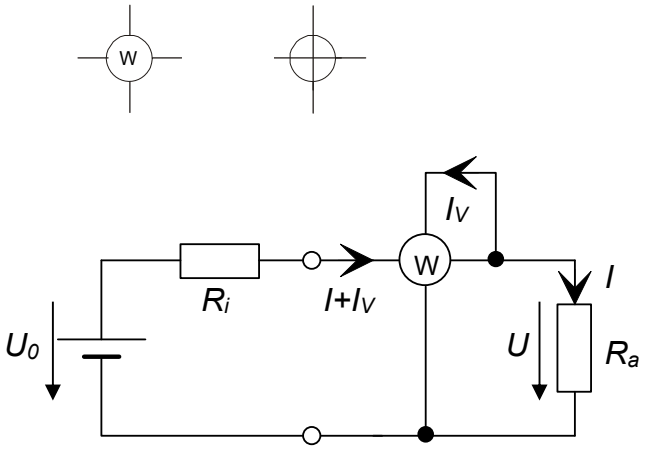


$I_V << I$ if $R_a << R_V$

Remark:
Modern state-of-the-art voltmeters have a comparatively high value of the voltmeter resistance R_V, so that usually the requirement for a voltage related measurement is fulfilled sufficiently.

Ohmmeter = Ammeter with an internal voltage source

Power meter



multiplying measuring system with 4 terminals:
- current path: low resistance
- voltage path: high resistance

Warning: Do not exceed
- neither the maximum permissible current of the current path (safe current)
- nor the maximum permissible voltage of the voltage path (safe voltage).

Classes of accuracy: 0,1 0,2 0,5 1 precision instruments; 1,5 2,5 5 industrial instruments

The numerical value means:

Failure in percent related to the final value of the measuring range.

Example:
For a "class 1" voltmeter, working in the 100 V measuring range, the failure is 1 V. This absolute failure of 1 V is valid also if only 5 V are measured in this 100 V range, i.e. then the relative failure is 20% ! Therefore measuring instruments should be used always in the lowest possible measuring range. This is essentially also valid for digital displays, where one cannot see directly the chosen measuring range.

2.5.2 Dropping resistor, shunt resistor

The indicating measuring system of a voltmeter or an ammeter requires power, it has a definite resistance R. Current and voltage are simultaneously at the terminals:

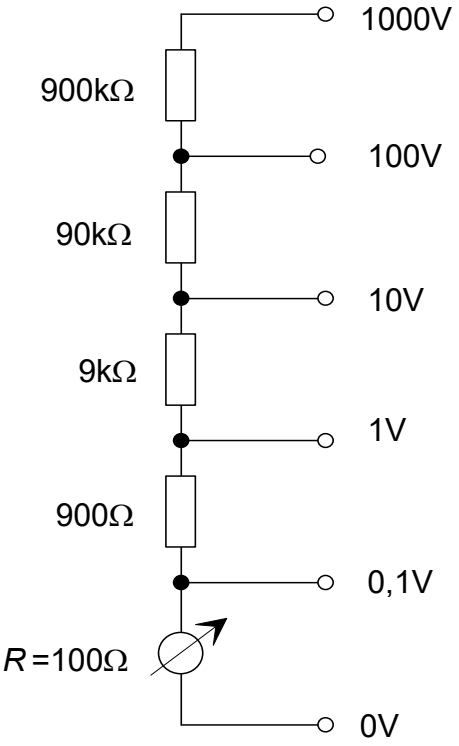
$\Rightarrow$ Measuring system can be used either as an ammeter or a voltmeter.

Example: We assume a measuring system with a resistance of $R = 100\ \Omega$.
Final value at full scale is for a current of $I = 1$mA, that is $U = 0{,}1$V

We now want to extent the possible measuring ranges of a voltmeter and an ammeter using the given measuring system. This can be achieved using additional resistors, either in series or in parallel.

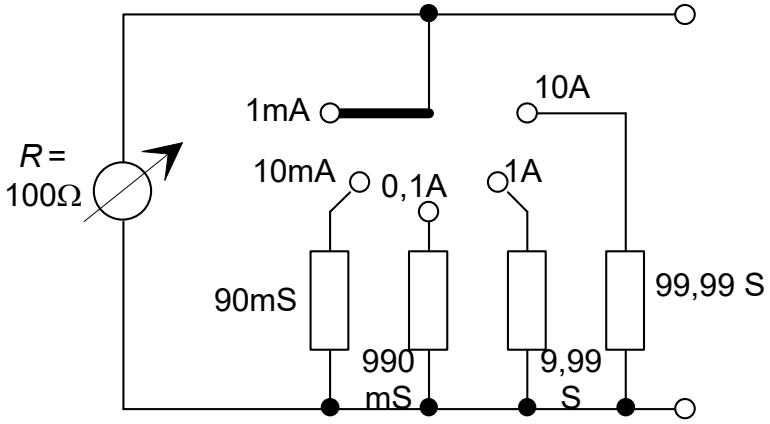
Voltmeter

Expansion of the measuring range by dropping resistors (multiplier resistors) in series



Ammeter

Expansion of the measuring range by shunt resistors in parallel



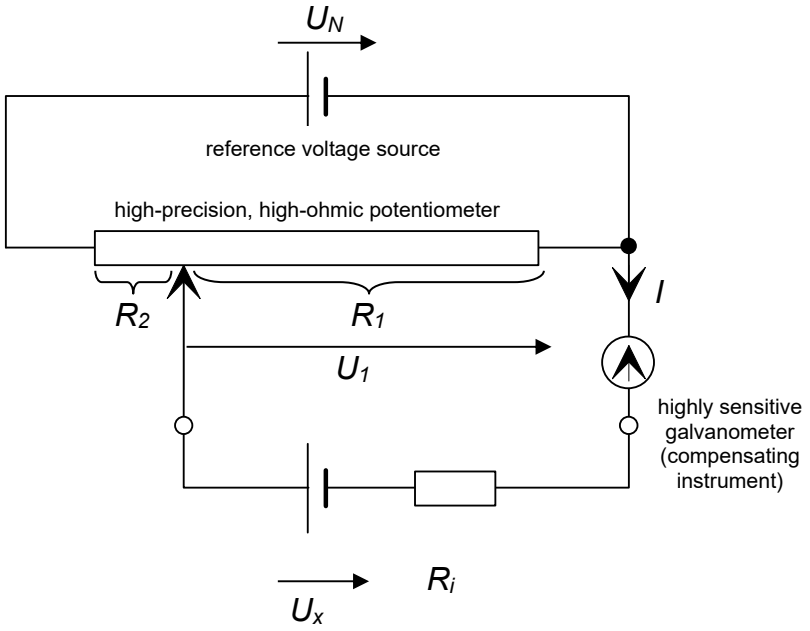
The voltmeter resistance usually is given in the following form:

$$\frac{\text{resistance}}{\text{final value of the measuring range}} = \ldots\ldots \frac{k\Omega}{V} \qquad \text{for example:} \quad \frac{100\ k\Omega}{V}$$

2.5.3 Compensating measurements

Using the compensating procedure, we can eliminate the influence of one electrical quantity on another, which should be measured. For example, a flowing current in a measuring circuit influences the measured voltage. In special cases, this is not acceptable. Here we want to get a current-free measurement of the voltages.

Examples are:
- weak power sources (high value of R_i)
- remote measuring (high resistance of the wires)
- precision measurements.



$I = 0$, if

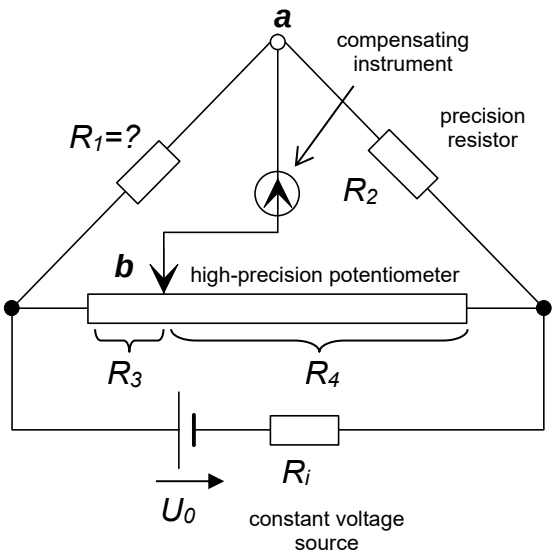
$$U_x = U_1 = \frac{R_1}{R_1 + R_2} \cdot U_N$$

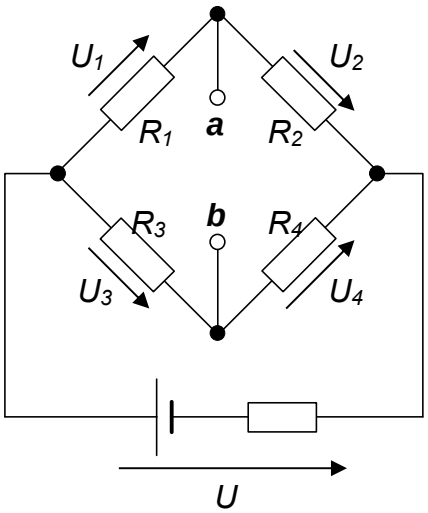
As I = 0, the resistances in the lower mesh (R_i, wires, etc.) have no influence, i.e. we do not have to considered them.

2.5.4 Bridge Circuits

A comparable procedure is used in case of bridge circuits.

Example: Wheatstone - Bridge for precision measuring of resistances.





In the case of an adjusted bridge no current flows through the current meter. Then the terminals **a** and **b** remain at the same potential even after a removal of the current meter. Both voltage dividers R_1, R_2 and R_3, R_4 then are similarily adjusted.

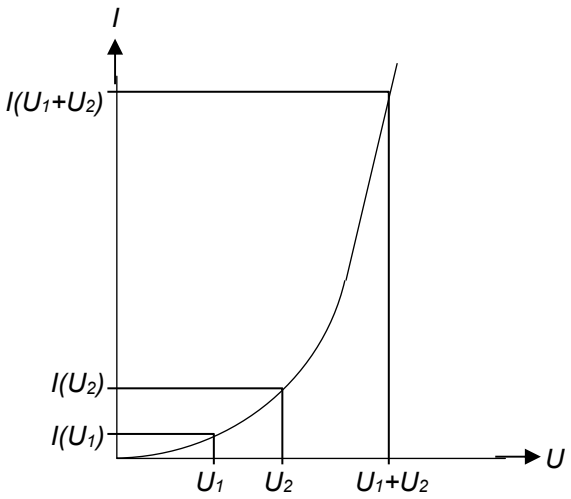
$$\frac{R_1}{R_2} = \frac{R_3}{R_4} \quad \Rightarrow \quad \boxed{R_1 = \frac{R_3}{R_4} \cdot R_2}$$

2.6 Networks with nonlinear elements

For nonlinear elements the principle of superposition is not valid:

$I(U_1 + U_2) \neq I(U_1) + I(U_2)$

Examples:
diodes, transistors,
gas discharge tubes, etc.



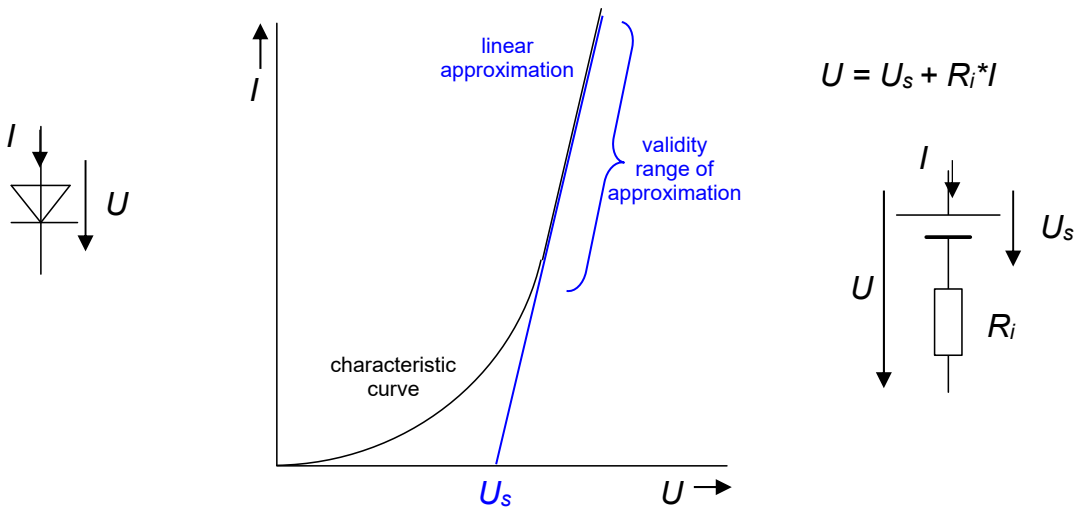
We have two general methods to investigate networks with nonlinear elements: (1) the nonlinear elements can be linearized in a limited range or (2) determination by a graphical method.

2.6.1 Linearization

Conditions: Operating point is known approximately.
Graph of the nonlinear element (characteristic curve) in this range is nearly linear.

Procedure: The nonlinear graph is approximated (i.e. replaced) by a straight line in the range of the operating point (see also chapter 2.1.3: linear approximation for $R(T)$). For more exact investigations also an approximation by a square parabola is possible.

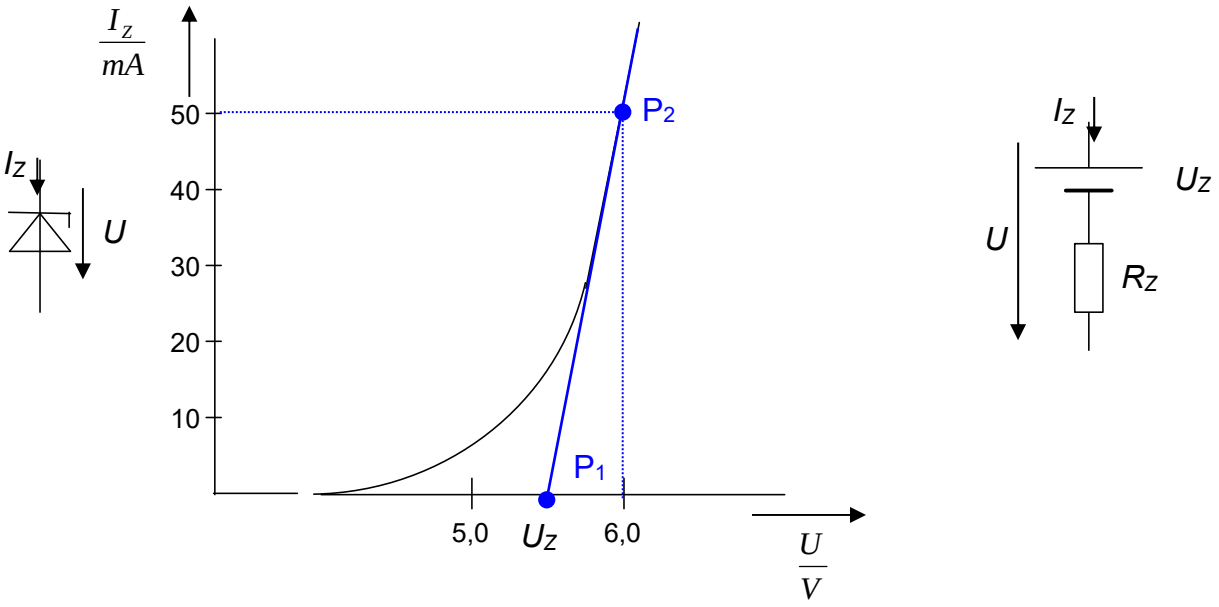
Example: Diode



We replace the diode by the equivalent circuit of a technical voltage source (ideal voltage source and an internal resistance R_i). U_s being the threshold voltage of the diode is identical to the open circuit voltage of the equivalent circuit.

Numerical example: Zener diode (Z diode) ZPD 5

Remark:
Zener diode is a diode with defined behaviour in reverse direction (defined breakdown voltage = zener voltage)



We assume, that the approximation and the equivalent circuit describe the response (behaviour) of the Z diode for I_Z > 10 mA (they are absolutely wrong for I_Z < 0 !)

Equation for the approximated straight line and the equivalent circuit:

2 points from the straight line: $U = U_Z + R_Z \cdot I_Z$

P1: $I_Z = 0$ $U = 5{,}5$ V $\rightarrow$ $\underline{U_Z = 5{,}5\ \text{V}}$

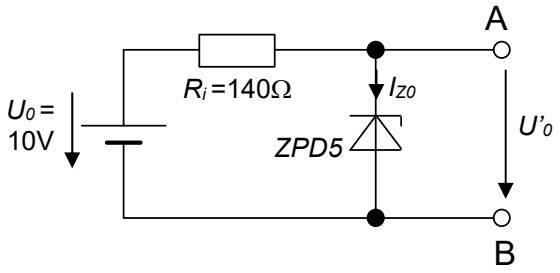
P2: $I_Z = 50$ mA $U = 6{,}0$ V $\rightarrow$ $6{,}0V = 5{,}5V + R_Z \cdot 50mA$

$\underline{R_Z = 10\ \Omega}$

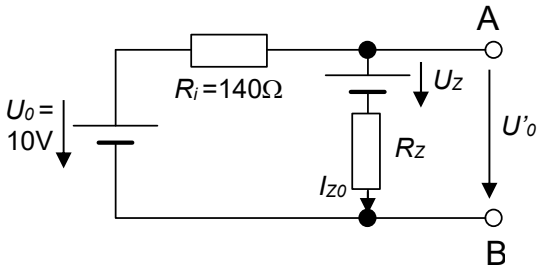
We continue this numerical example and investigate the behaviour of a circuit containing a ZPD 5 with or without load.

Circuit without load:

original circuit



Z diode replaced by voltage source

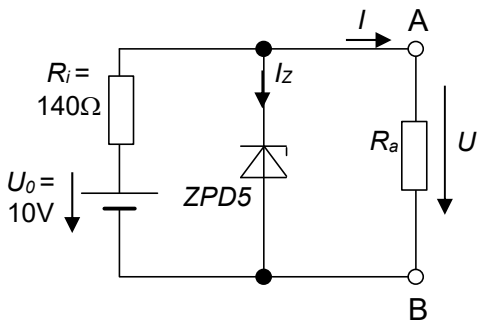


$$I_{Z0} = \frac{U_0 - U_Z}{R_i + R_Z} = \frac{10V - 5.5V}{140\Omega + 10\Omega} = \frac{4.5V}{150\Omega} = 30mA$$

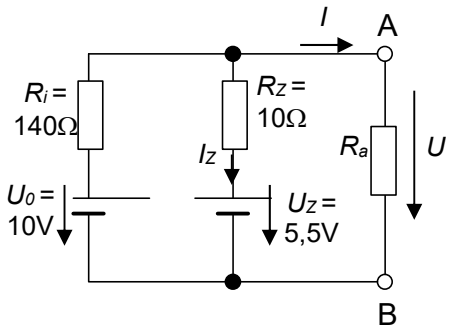
Because of I_Z > 10mA, the equivalent circuit is valid (counter example: R_i = 2,2kΩ)

Response of the circuit with load:

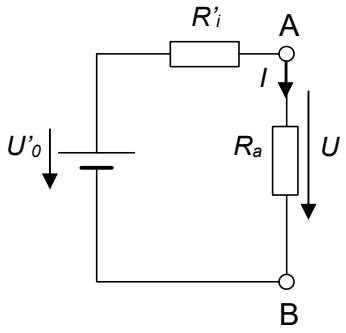
original circuit



Z diode replaced by voltage source



equivalent voltage source



$$U_0^{'} = U_Z + R_Z \cdot I_{Z0} = 5{,}5V + 10\Omega \cdot 30mA = 5{,}8V$$

$$R_i^{'} = \frac{R_i \cdot R_Z}{R_i + R_Z} = \frac{140\Omega \cdot 10\Omega}{140\Omega + 10\Omega} = 9{,}33\Omega$$

The characteristic curves for this circuit can be described:

- without zener diode ZPD 5: $U = U_0 - R_i \cdot I = 10V - 140\Omega \cdot I$
- with zener diode ZPD 5 $U = U_0^{'} - R_i^{'} \cdot I = 5{,}8V - 9{,}33\Omega \cdot I$

Graphical description: see lecture !

Finally we determine the validity range of this equivalent circuit with ZPD 5:

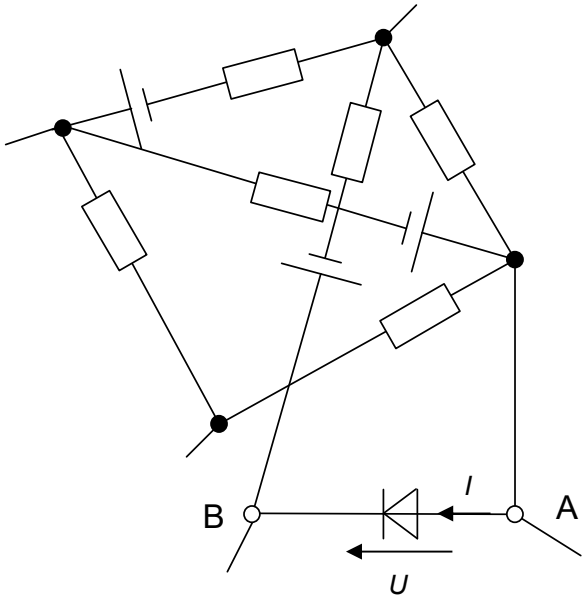
$$I_Z > 10mA$$

$$U = U_Z + R_Z \cdot I_Z \quad \Rightarrow \quad U > 5{,}5V + 10\Omega \cdot 10mA \quad \Rightarrow \quad U > 5{,}6V$$

$$I = \frac{U_0^{'} - U}{R_i^{'}} \quad \Rightarrow \quad I < \frac{5{,}8V - 5{,}6V}{9{,}33\Omega} \quad \Rightarrow \quad \underline{I < 21{,}4mA}$$

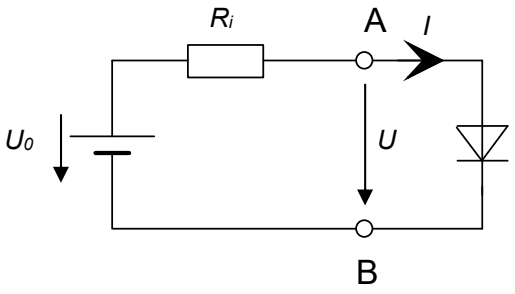
2.6.2 Graphical determination method of operating points

Assumption: A complex network consists of an arbitrary number of linear elements. It further contains **exactly one** nonlinear element. If there is another nonlinear element, this method is not possible.



We apply the well-known method of an equivalent voltage (or current) source:

- Take out the nonlinear element
- Determine the equivalent voltage source of the rest of the circuit
- Load the equivalent voltage source with the nonlinear element

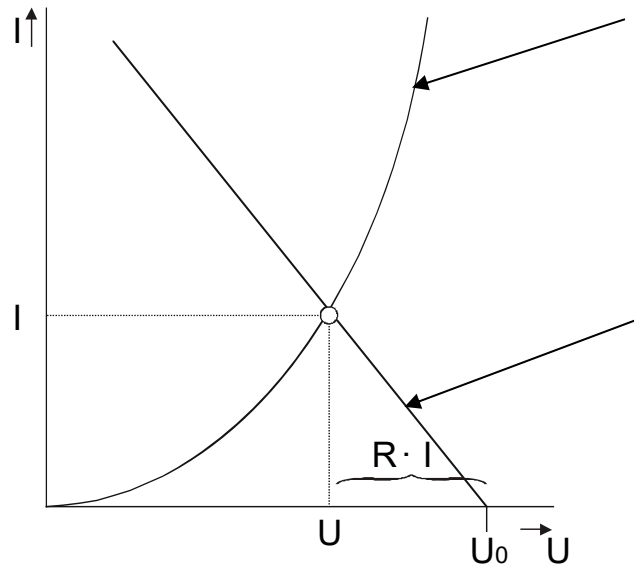


Equivalent voltage source with load:

$$U_0 = R_i \cdot I + U$$

$$U = U_0 - R_i \cdot I$$

I(U) - relation of the nonlinear element is given in a graphical form.



Curve 1:
This relationship between *U* and *I* is always valid, independent to which source the nonlinear element is connected
⇒ determination by the nonlinear element.

Curve 2:
This relationship between *U* and *I* is always valid, independent how the source is loaded
⇒ determination by the source

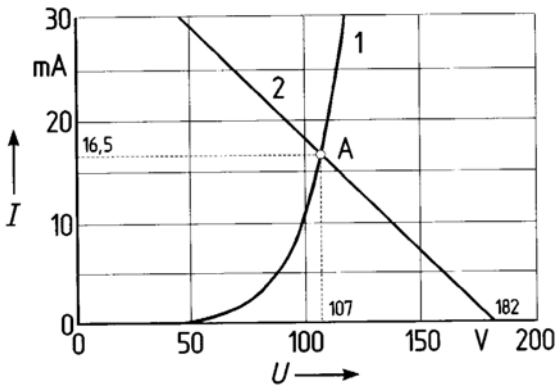
$$U = U_0 - R_i \cdot I$$

(equivalent voltage source with load)

Both relationships have to be fulfilled simultaneously, so the coordinates of the intersection point deliver the solution.

The advantage of this method is that we do not need to know the operating point in advance (see the problem of the validity range using the linearization method).
The disadvantage is that for this method only one nonlinear element is possible.

Additional numerical example: Coordinates of the operating (intersection) point [1]



Curve 1:
Nonlinear element

Curve 2:
Source with U_0 = 182V, R_i = 4,55kΩ

⇒ $U = 182V - 4{,}55k\Omega \cdot I$

Intersection point:

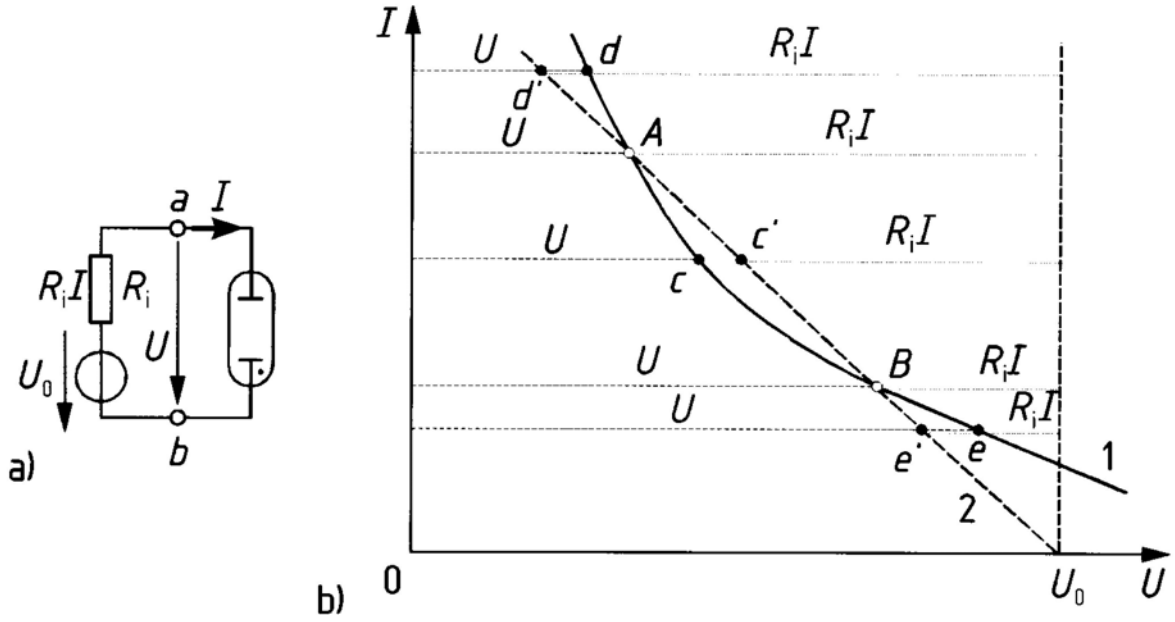
U = 107V, I = 16,5mA

2.6.3 Stability investigations

We have to use the graphical determination method.

There are applications where the current still increases while the voltage decreases (example: gas discharge tubes, air discharge, NTC resistors). In these cases stabilization is necessary, for example by a resistor in series.

Example [1]:



Curve 1: *I(U)*-relation of the gas discharge tube
Curve 2: Operating graph (straight line) of the equivalent voltage source U_0, R_i

Point B (unstable):

1) If the current *I* decreases slightly (B → e, e'), the voltage at the discharge path of the gas discharge tube increases more (e), than the voltage drop $R \cdot I$ within the voltage source decreases (e'):

$$U(e) + R \cdot I(e') > U_0$$

The circuit needs more voltage, than the voltage source delivers →
moving process away from B → current decreases further, arc goes out !

2) If the current *I* increases slightly (B → c, c'), the voltage at the discharge path decreases more (c), than the voltage drop $R \cdot I$ increases (c'):

$$U(c) + R \cdot I(c') < U_0$$

This leads to a further current increase, until point A is reached →
moving process away from B → point A

Point A (stable):

1) If the current *I* increases slightly (A $\rightarrow$ d, d'), the voltage $R \cdot I$ increases more (d'), than the voltage at the discharge path decreases (d):

$$U(d) + R \cdot I(d') > U_0$$

The circuit needs more voltage, than the voltage source delivers $\rightarrow$
moving process back to A

2) If the current *I* decreases slightly (A $\rightarrow$ c, c'), the voltage $R \cdot I$ decreases more (c'), than the voltage at the discharge path increases (c):

$$U(c) + R \cdot I(c') < U_0$$

This leads to a current increase, until point A is reached again $\rightarrow$
moving process back to A

Using these results, we can formulate finally the stability conditions for this example. Stability is given:

- if a current increase leads to an increase of the sum voltage $U + R \cdot I$ (to more than U_0);
- if a current decrease leads to a decrease of the sum voltage $U + R \cdot I$ (to less than U_0).

$$\frac{d(U + R \cdot I)}{dI} > 0 \qquad \Rightarrow \qquad \underline{\frac{dU}{dI} + R > 0}$$

$$R > -\frac{dU}{dI} \qquad \text{or} \qquad \frac{1}{R} < -\frac{dI}{dU}$$

For a stable operating point the trailing graph of the nonlinear element in the *I-U*-diagram given above has to be steeper in this operating point than the graph (characteristic curve) of the active source (voltage or current source).

3. Electrostatic field

3.1 Charges as the cause of electric fields

3.1.1 Effects of charges

Here we are referring to the observation made in section 1.3 ‘Charging through friction’.

There are some insulators to which electrons have a higher affinity (e.g. hard rubber) than to others (e.g. a cat’s fur). If two materials with different electron affinities are brought into contact to each other, electrons will move from one insulator to the other. Positive ions will stay with the insulator which has less electron affinity.

triboelectric series

very dry hand	positive (low electron affinity)
leather	
glass	
hair	
wool	
fur	
silk	
paper	
cotton	
hard rubber	
polyester	
pvc	negative (high electron affinity)

If two of these materials are rubbed against each other, the material with lower electron affinity will lose electrons and become positively charged. The material with higher electron affinity will gain electrons and become negatively charged. In the list above the “distance” between two materials shows how strong the charge is. Two materials with a greater distance will be charged stronger.

Example 1: Hard rubber rod and fur are rubbed together. Fur has lower electron affinity, it will lose electrons. The leftover protons without partners cause the “positive charge”, it means that fur becomes positively charged. The rubber rod has higher electron affinity, “negative charge” will land on it. It means the rubber rod will gain electrons and becomes negatively charged.

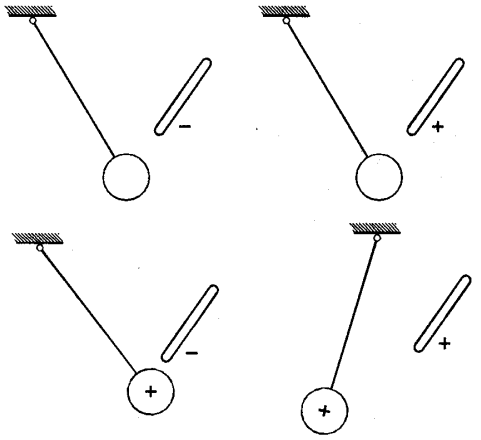
Example 2: Glass rod and silk are rubbed together. Here, the glass becomes positively charged and, respectively, the silk negatively. Some of the glass’s electrons were transferred to the silk through rubbing.

If one rubs a glass rod with a woollen cloth or a piece of fur, the observation can be made that afterwards the glass rod attracts light objects such as feathers or shreds of paper. The space around the charged rods is transposed into a “special condition” through the presence of the charges. It has the physical characteristic to exert

influence over particular objects that either attracts them to or repels them from the rod. In physics, such a space with special characteristics is termed a field.

Conclusion: Electrically charged bodies are surrounded by electric fields.

Based on the described effects, this field is a force field. The next figure shows an experimental configuration, from which we can draw some fundamental conclusions. A small, light sphere which has been metallised on the outside is hung on a thin, insulating thread. In the top half of the picture, the sphere is not charged, or neutral. In the bottom half, the sphere is positively charged.



When the above-shown rod made of glass (positively charged) nears the uncharged sphere, its attraction to the rod can be observed. The same effect can be achieved by rubbing a hard rubber rod (negatively charged) with a piece of fur. We now touch the sphere with one of the charged rods and notice that it is now being repelled from the rod which touched it and, in contrast, attracted by the other rod.

While the attraction of the uncharged sphere will be explained in depth in the chapters about induction and polarization, the explanation of the second observation of the charged sphere is not difficult:

During the contact of the sphere with the rod, a part of the rod’s electric charge flows onto the sphere. The sphere then carries the same type of charge as that particular rod. The other rod apparently carries the opposite charge, since the opposite force is emitted from it.

Agreement:
A lack of electrons is identified as positive. An excess of electrons is identified as negative. With this agreement of positive and negative charge, it can be proven by further experiments that the glass rod is positively charged and the hard rubber rod is negatively charged.

Conclusion: differently signed charged objects attract each other, same signed objects repel each other

3.1.2 Charge Conservation

In a further experiment, the rod is rubbed again. It is then covered with a cloth and brought near the sphere: nothing happens. But if the cloth and the rod then are separated, both of them show a distinct effect.

Through experiments like the one above, it can be displayed that charges are of a contrary nature. It becomes clear that with the creation of one type of charge, the other is created simultaneously. We describe this phenomenon with the conservation law of electric charges:

The sum of all created positive and negative electric charges is always equal to zero.

According to this law, it becomes obvious, that there is a contrary force effect of both types of charges. Therefore it is useful to identify them as positive or negative.

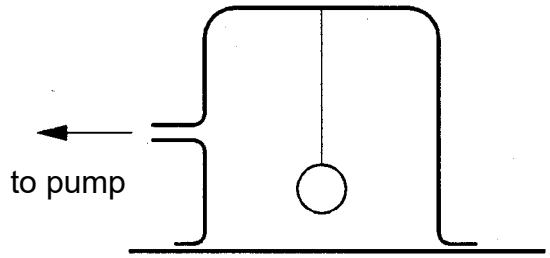
The experiment in 3.1.1 shows that the charge can be transferred from one body to another. Now it must be examined whether the different bodies behave differently from an electrical point-of-view.

If we touch an electrically charged metallic sphere with a finger, we see that the metallic sphere is subsequently completely nonelectric. If the rod, however, is touched, it becomes nonelectric only on the spot where it was touched. During the contact with the finger the electric charge is transferred to the finger. Apparently, all of the charged particles on the metallised sphere can move to the point of contact and flow off of the sphere. This is, however, not possible for the rods. Thus the bodies are classified either as conductors or insulators, depending on whether the charge spreads out over the entire body in a very short time, or if they stick to the point where they were created by rubbing. There is no absolute non-conductor. All bodies are more or less (good) conductors.

All experiments have been conducted in air. Therefore the question arises:
‘To what extent is the electric field associated with the medium of air?’

To answer this question, the experiment from 3.1.1 is repeated and this time the sphere is placed in a vacuumized glass container. When a charged rod is brought close to the container, the same force effects as in air can be observed without noticeable differences. The field around the charged rod penetrates the glass wall and is present in the vacuum as well. Insulators can be placed between the sphere and the rod, but each time the force effects can be observed. A force effect cannot, however, be observed when the sphere is surrounded by a metal housing.

Conclusion:
An electric field can exist in non-conductors or insulators. No electrostatic field is possible in conductors. This leads to an immediate balance in charge.



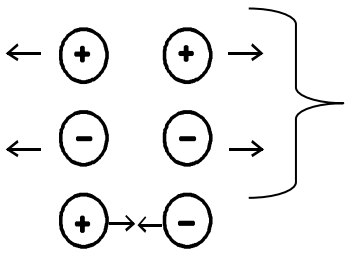
Difference between conductor and non-conductor:

- Isolator (non-conductor): 10^7 … 10^9 free electrons / cm^2
- Semi-conductor 10^{10} … 10^{18} free electrons / cm^2
- Electrical conductor: approx. 10^{23} free electrons / cm^2

3.2 Forces between electric charges, Coulomb's law

3.2.1 Forces between two charges

We summarize from 3.1: There are two types of electricity.



same charges: repulsion

opposite charges: attraction

Charge: "Quantity of electricity", evades sensual perception.

Unit: $[Q] = 1C = 1A \cdot s \quad (= 1A \cdot 1s)$

Elementary charge: $e = 1{,}6 \cdot 10^{-19}\,C$ (smallest known quantity of charge)

Electron: $Q_{el} = -e$

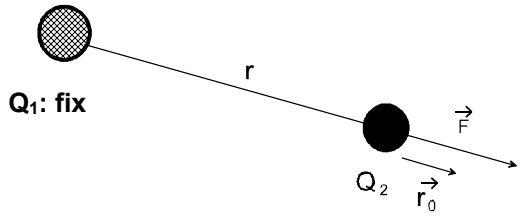
Proton: $Q_{\mathrm{Pr}} = +e$

Quantitative law of forces: Coulomb's law

The force effect between two charges can be described by a simple law.

Ideal case: 2 point charges in space

Q_1, Q_2, distance r $\quad |\vec{r}_0| = 1$



$\vec{r}_0$: *unit vector*

Conclusions:

1) $\vec{F}$ lies in connecting line Q_1 to Q_2
2) $F \sim Q_1 \cdot Q_2$ (sign)
3) $F \sim \frac{1}{r^2}$

Relation: $\vec{F} = p \cdot Q_1 \cdot Q_2 \cdot \frac{1}{r^2} \cdot \vec{r}_0$ with $p = \text{factor of proportionality} = \frac{1}{4 \cdot \pi \cdot \varepsilon_0}$

$$\varepsilon_0 = \frac{1}{36 \cdot \pi \cdot 10^9} \frac{C^2}{N \cdot m^2} = 8{,}854 \cdot 10^{-12} \frac{F}{m}$$

ε_0 : electric constant (vacuum permittivity)

Coulomb's law (law of forces)

$$\boxed{\vec{F} = \frac{1}{4 \cdot \pi \cdot \varepsilon_0} \cdot \frac{Q_1 \cdot Q_2}{r^2} \cdot \vec{r}_0}$$

Analogous observation:

Electrostatic	**Gravitation**
two electric charges attract each other (or repel each other)	two bodies (sun, planets, moons) attract each other
$F \sim \frac{Q_1 \cdot Q_2}{r^2}$	$F \sim \frac{m_1 \cdot m_2}{r^2}$

both formulas are valid for spherical structures

3.2.2 Electrostatic field of a point charge

Newton's Law of Gravitation and Coulomb's Law are useful, if two balls or two charges of comparable dimensions are present.

For the description of the attraction between the earth and an object on the earth, it is more convenient to state, that the object is located within the gravitational field of the earth.

Procedure: One of the bodies is assigned a gravitational field. The other is located within this gravitational field, so that gravity acts upon it.

$$F = \underbrace{\gamma \cdot \frac{m_1}{r^2}}_{g} \cdot m_2$$

$$F = g \cdot m_2$$

Earth:

$$\gamma = 6{,}67 \cdot 10^{-11} \frac{m^3}{kg \cdot s^2}$$

$$m_1 = 5{,}98 \cdot 10^{24}\, kg$$

$$r = 6370 km$$

$$g = 9{,}81 \frac{m}{s^2} \text{ (on the earth's surface)}$$

Now we formulate a similar procedure for the description of forces between two electrical charges:

Procedure: One of the charges produces an electrical field. The other charge is located within this electrical field, so a force acts upon it:

$$F = \underbrace{\frac{1}{4 \cdot \pi \cdot \varepsilon} \cdot \frac{Q_1}{r^2}}_{E} \cdot Q_2$$

$$F = E \cdot Q_2$$

$$\boxed{E = \frac{Q}{4 \cdot \pi \cdot \varepsilon \cdot r^2}}$$

Electrical field of a point charge

3.2.3 Electrical charges in an electrical field, Electrical field strength

Proof of a gravitational field through force acting upon a mass.

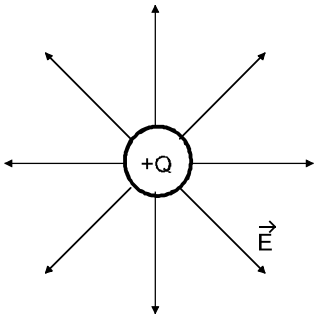
Direction of gravity = direction of the gravitational field

$$\vec{F}_g = m \cdot \vec{g}$$

Proof of an electrical field through force acting upon an electrical charge.

Direction of electrical force = direction of the electrical field

$$\boxed{\vec{F} = Q \cdot \vec{E}}$$



Example: Positive point charge with electrical field $\vec{E}$

A charge Q_2 located in this electrical field experiences the force.

$$\vec{F} = Q_2 \cdot \vec{E}$$

For $\mathbf{Q_2} > 0$, $\vec{F}$ has the same direction as $\vec{E}$

$\Rightarrow$ Repulsion

For $Q_2 < 0$, $\vec{F}$ has the opposite direction as $\vec{E}$

$\Rightarrow$ Attraction

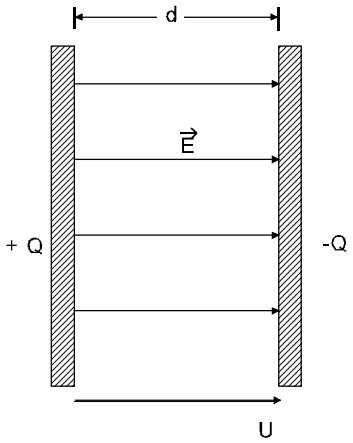
Physical unit of electrical field strength:

$$\text{Unit}\left[E\right] = \frac{\text{Unit}\left[F\right]}{\text{Unit}\left[Q\right]} = \frac{N}{As} = \frac{Nm}{As \cdot m} = \frac{Ws}{As \cdot m} = \frac{VA}{Am}$$

$$\underline{\text{Unit}\left[E\right] = \frac{V}{m}}$$

Electrical field strength = Potential differential

simplest case (homogeneous Field):



$$E = \frac{U}{d}$$

$$\underline{U = E \cdot d}$$

in general:

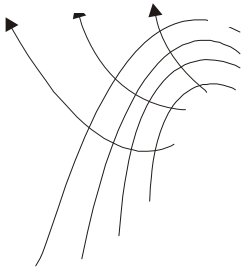
$$\vec{E} = \frac{dU}{d\vec{s}}$$

with $\vec{E}$ and $d\vec{s}$ in the direction of the greatest slope

$$\boxed{U = \int \vec{E} \cdot d\vec{s}}$$

(trajectories) field lines

(contour lines) Equipotential lines



3.2.4 Superposition principle

If several single charges are present, each creates its own electrical field and all partial fields overlap linearily to a total field. This is valid because the relationship between the field strength (*E*) and the charge (*Q*) that it causes is linear.

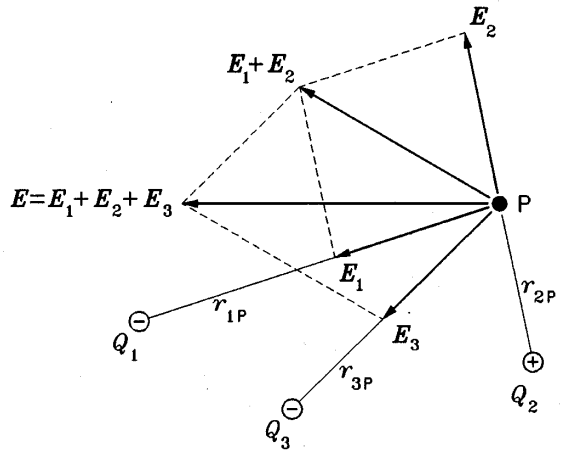
This procedure for three different charges is depicted in the next picture. P is the so-called test point in which the field strength should be determined. The quantities of the various field strengths in P are shown in the equation:

$$|E| = \frac{1}{4\pi \cdot \varepsilon} \cdot \frac{Q_i}{r_{iP}^2}; \qquad i = 1,2,3$$

The directions correspond to the directions of the charges' connecting lines to the test point. The partial fields can be determined using the known distances and single charges. Through vectorial addition of the individual fields results a complete field [1, 5]:

$$\vec{E} = \vec{E}_1 + \vec{E}_2 + ... + \vec{E}_n$$

Important: The whole space is homogenous, which means the permittivity number is the same (3.5.4).

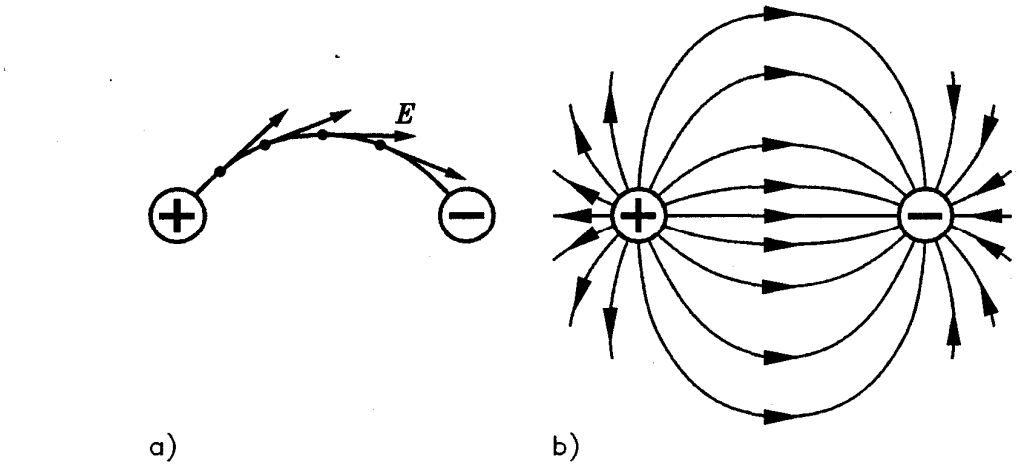


3.2.5 Field lines and field line diagrams

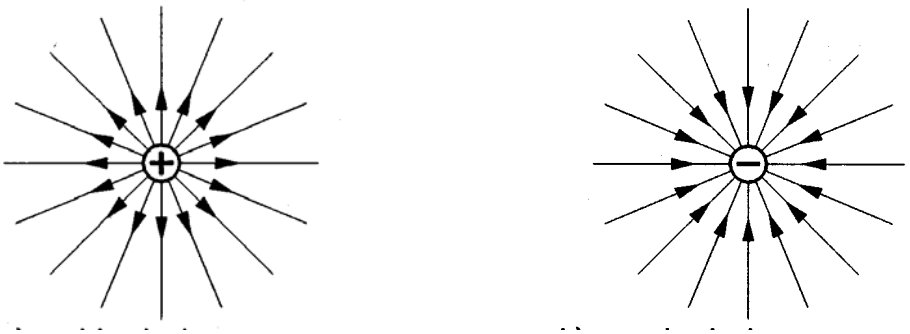
Electric fields are indicated as field lines between charges, creating an optical impression of the structure of the electrical field.

The diagrams illustrate that the field lines always have a starting point and an ending point. According to our definition, they begin at the positive charge and end at the negative charge [5]. Thus, it is deemed:

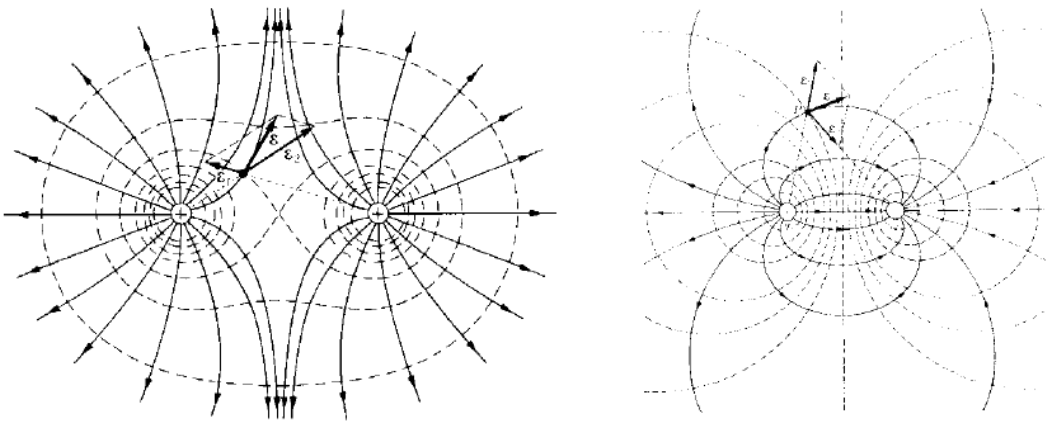
The sources and sinks of the electrical field lines are the positive and negative electric charges.



The next two diagrams show the field lines near a positive and near a negative charge. What we see is that the field lines flow in a radial direction. These lines point away from a positive charge, whereas these lines point toward a negative charge [5].



Further diagrams:



Charges with same sign **Charges with opposite sign**

3.3 Idealized charge distribution

In the next chapters we will be working with various charge distributions, but first a short summary of the most important distributions and their calculations.

3.3.1 Point charge

At a distance which is large in comparison to all three dimensions of a space charge area, the effect can be adequately described by using an ideal point charge in the centre of the space charge area. As the name says, the complete charge of the space charge area is concentrated in one point:

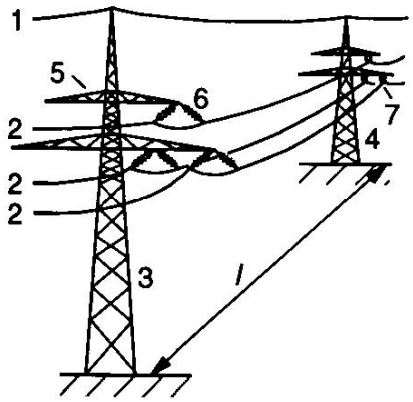
A point charge is a body with a finite charge Q but negligible small volume.

In a borderline case, $r \rightarrow 0$ so that the space charge density of a point charge is infinitely large. We were introduced to the point charge in 3.2.

3.3.2 Line charge

If the distance between an observer and a space charge area is relatively large in two directions, the effect of such areas can often adequately be described with the assistance of an ideal line charge.

An example of a line charge is the high voltage power line [1]:



1) earth (ground) cable
2) conducting cable
3) anchor pylon
4) voltage tower (support pylon)
5) cross bar (traverse)
6) horizontal insulator
7) hanging insulator
l) field length

At a distance from the power line that is large in comparison to the diameter of the power line, the charge of this power line can be understood as a line charge. This is a linear distributed charge.

A homogenous or a location dependent charge distribution can exist along a line.

The line charge density $\lambda(x)$ is a local scalar quantity this means each point of the line has been allocated a scalar value. The total charge in any line with the length *l* is then calculated using the known line charge density $\lambda(x)$:

$$Q(l) = \lim_{\Delta l \to 0} \sum_{\Delta l} \lambda(x) \cdot \Delta l$$

We assume a line length *l* is divided into very small line segments Δl, and their charges $\Delta Q = \lambda \cdot \Delta l$ are summed up. Thereby the line charge density is homogenous

within a line segment. Going to infinite small line segments the sum merges in the integral:

$$Q(l) = \int_l \lambda(x) \cdot dl$$

In the common case of a homogenous line charge density λ_0, the total charge results from the product of the line charge density λ_0 and the sum of line segments, which is the length *l*:

$$Q(l) = \lambda_0 \cdot l$$

Unit of line charge density: $[\lambda] = \frac{[Q]}{[l]} = \frac{As}{m}$ „ charge per length “

3.3.3 Area charge

If the expansion of a space charge area in a dimension is very small, the effect of this area can be sufficiently described with assistance of ideal area charges.

Here, the charge distribution of a metal sphere serves as an example for an area charge. Thus:

- same signed charges repel each other;
- the charges drift apart to the surface of the sphere because of conductivity;
- for symmetrical reason, the area charge is homogenous (no preferred direction).

When two charged spheres are brought into close proximity, the charges will influence each other and the area charges become inhomogeneous:

- **repulsion of same signed charges;**
- **attraction of different signed charges.**



The area charge density $\sigma(x,y)$ is a local scalar quantity, this means each point of the area has been allocated a scalar value. The total charge in any area *A* is calculated using the known area charge density $\sigma(x,y)$:

$$Q(A) = \lim_{\Delta A \to 0} \sum_{\Delta A} \sigma(x,y) \cdot \Delta A$$

We assume the area *A* is divided in very small partial areas ΔA, and their charges $\Delta Q = \sigma \cdot \Delta A$ are summed up. Thereby the area charge density is homogenous within a partial area. On the border crossings of infinite small partial areas the sum merges in the integral:

$$Q(A) = \iint_A \sigma(x,y) \cdot dx \cdot dy = \int_A \sigma(x,y) \cdot dA$$

In the common case of a homogenous area charge density $\sigma(x,y) = \sigma_0$, the total charge results from the product of the area charge density σ_0 and the sum of partial areas, which is the area *A*:

$$Q(A) = \sigma_0 \cdot A$$

Unit of area charge density: $[\sigma] = \frac{[Q]}{[A]} = \frac{As}{m^2}$ „ charge per area “

3.3.4 Space charge

Point charge, line charge, and area charge are idealized charge distributions that do not occur in nature for energetic reasons. In practice, however, they are helpful because they are easy to deal with mathematically and deliver adequately accurate results in many cases.

In reality space only charges actually occur. For example, the space charge zones in many electronic components (e.g. pn-diode) affect their behaviour in a decisive way in reference to the clamps.

The space charge density $\rho(x,y,z)$ is a local scalar quantity, this means each spatial point has been allocated a scalar value. The total charge in any volume *V* is calculated using the known space charge density $\rho(x,y,z)$:

$$Q(V) = \lim_{\Delta V \to 0} \sum_{\Delta V} \rho(x,y,z) \cdot \Delta V$$

We assume the volume *V* is divided into very small partial volumes ΔV and their charges $\Delta Q = \rho \cdot \Delta V$ are summed up. Thereby the space charge density is homogenous within a partial volume (e.g. equal to the space charge density in the centre of the partial volume). On the border crossings of infinite small partial volumes the sum merges in the integral:

$$Q(V) = \iiint_V \rho(x,y,z) \cdot dx \cdot dy \cdot dz = \int_V \rho(x,y,z) \cdot dV$$

In the case of a homogenous space charge density $\rho(x,y,z) = \rho_0$, the total charge results from the product of the space charge density ρ_0 and the sum of the partial volumes, which is the volume *V*:

$$Q(V) = \rho_0 \cdot V$$

Unit of space charge density: $[\rho] = \frac{[Q]}{[V]} = \frac{As}{m^3}$ „ charge per volume “

3.4 Movement of a charge in an electrical field

3.4.1 Work, potential, voltage

In an electrical field, a force is exerted on a charge. An outside energy is necessary to move a charge in an electric field against its natural moving direction; it means work is required. If the shift is taking place in the same direction as the field, then the field performs the necessary work.

In each situation, the expended energy must be calculated. It is positive when it is performed by the field. It is negative when it is externally supplied. Work is a scalar quantity. As is known from mechanical science, it is defined as the product of the acting force and the distance travelled:

$$\Delta W = |F| \cdot \Delta s$$



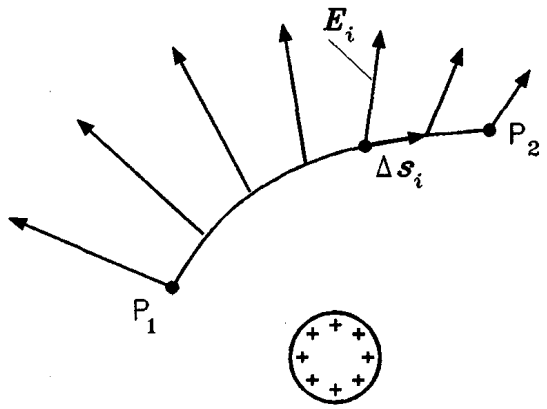
The direction of the charge’s movement and the force’s direction are variable. In the example on the left, the orientations of force and path are the same. The accomplished work ensues as a simple product of force times path length. In the example on the right, the orientation of the force is different than the orientation of the path. Only that portion of the force which is oriented in the direction of the movement contributes to the energy. This portion can be calculated using the following formula:

$$\Delta W = |F| \cdot \Delta s \cdot \cos\alpha$$

This equation can be reformulated when path length is interpreted as a vector. In this case, externally supplied energy is calculated as a scalar product of the vector quantities force and distance (scalar product). If one takes into account that the force is equal to the product of charge and electric field strength, it ensues that the externally supplied energy is also a scalar product of the electrical field strength and the distance (multiplied by the scalar quantity charge, of course):

$$\Delta W = \left|\vec{F}\right| \cdot \Delta\vec{s} = Q \cdot \vec{E} \cdot \Delta\vec{s}$$

As depicted in the next diagram, neither direction nor magnitude of the electric field strength along the path is constant. To calculate the energy the path length has to be divided in very small segments, in which the field force can be assumed constant [1].



The total energy results from the sum of the individual vector products:

$$W = \sum_{i=1}^{n} \Delta W_i = Q \cdot \sum_{i=1}^{n} \vec{E}_i \cdot \Delta\vec{s}_i$$

When the individual path segments become infinitely small, the summation becomes an integration again:

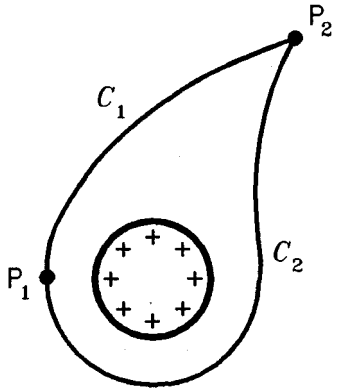
$$W = Q \cdot \int_{P_1}^{P_2} \vec{E} \cdot d\vec{s}$$

This integration is to be conducted along a certain path. The integral is therefore identified as the line integral of the field strength along a path. As depicted in the next diagram with a positively charged sphere, two different paths are possible between point P_1 and P_2. These paths are labelled C_1 and C_2.

The question is whether the calculated energy depends on the chosen path. It is assumed that the charge of the sphere does not change. Furthermore we assume that after one lap around the circuit, upon return to the start, the field gets its original state. This means, however, that all the work during the lap has a value of zero. This is only possible when the work performed is independent from the path chosen.

$$\int_{P_1}^{P_2} \vec{E} \cdot d\vec{s}_1 = \int_{P_1}^{P_2} \vec{E} \cdot d\vec{s}_2$$

(c_1) (c_2)



Upon further consideration, this also means that an integration of the electric field along a closed path must be equal to zero [1]:

$$\oint \vec{E} \cdot d\vec{s} = 0$$

The circle of the integral sign indicates the closed path. In general, a field whose line integral along a closed path equals zero is identified as a vortex free field. This also means that there are no self-contained field lines. Since a field force always has the same direction on a field line, the integral would result in an infinite value, but not zero. In this regard, the electric field differs fundamentally from the magnetic field.

Electrostatic fields don't have self-contained closed field lines.
They are vortex free.

Since, like concluded, the result of the integration is independent from the path, the path can be chosen in a way, that the integration can be accomplished as easily as possible.

No matter which path is chosen, the result for the energy depends only on the start and end of the path. Each point in the field can therefore be allocated a characteristic function value. This function value is identified as the electrostatic potential. It is not predetermined. The relative quantity is defined by the following equation:

$$\int_{P_1}^{P_2} \vec{E} d\vec{s} = -(\varphi(P_2) - \varphi(P_1))$$

Any constant can be added to or subtracted from the electrostatic potential without changing the result of the line integral. In general, we choose the potential of the earth or the potential of the infinitely distant space to zero. If the electrostatic potential in every point of the space is known, then the field can be clearly determined. Instead of the field vector *E*, the scalar quantity φ can be used to characterize the electrostatic field. If the points P_1 and P_2 are very close to each other, then the above equation changes to:

$$\vec{E} \cdot d\vec{s} = -d\varphi \qquad \text{or} \qquad \vec{E} = -\frac{d\varphi}{d\vec{s}}$$

The field force can therefore be calculated clearly from the electrostatic potential. The potential difference between P_1 and P_2 is also identified as electric voltage.

As a result, it can now be concluded:

- The work which the field performs by shifting a charge *Q* from P_1 to P_2 is equal to the product of the electric charge and the voltage between points P_1 and P_2.

- Accordingly, the physical meaning of the term potential can be interpreted. If a positive test charge is shifted in an electric field from zero potential to a higher potential, it will absorb energy. This energy is identified as potential energy. Thus the potential energy of a charge in the field is (based on the zero potential) equal to the product of potential and charge.

Voltage and potential are useful and essential terms in the static electric field. They indicate which work the field is able to perform per charge unit. Hence, a separate

designation for the unit of voltage and potential was introduced, the volt (V). This unit is defined as follows:

$$[W] = [Q \cdot U] = 1V \cdot C = 1V \cdot As = 1Nm = 1kg \cdot \frac{m^2}{s^2}$$

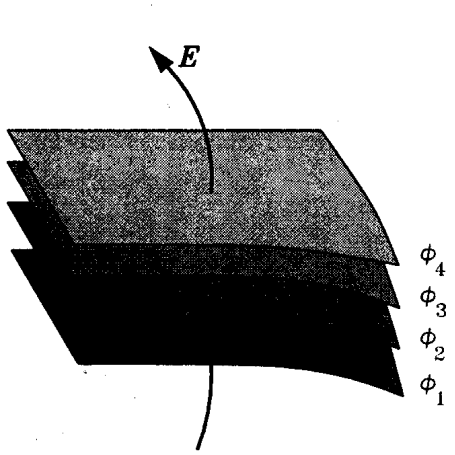
The result, in basic units, is: $1V = 1\frac{kg \cdot m^2}{A \cdot s^3}$

Once the unit for voltage is known, the unit for electric field strength can be mentioned again. According to our results, the unit of the electric field strength is equal to the unit of the voltage divided by the unit of the length.

Thus ensues: $[E] = \frac{[U]}{[l]} = \frac{V}{m}$

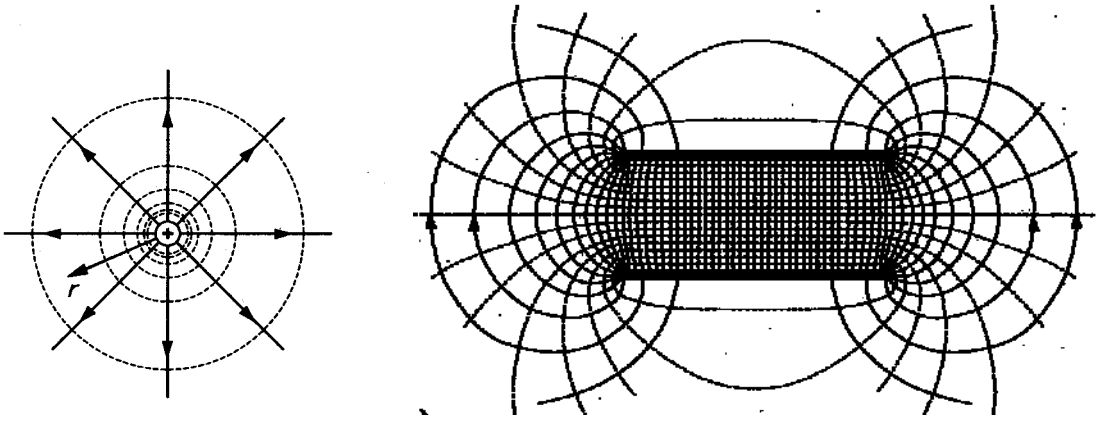
3.4.2 Equipotential areas

Finally, some depictions of potential are shown in this chapter. Areas where the potential is constant are identified as equipotential areas. The neighbouring picture shows examples of such equipotential areas [1].



If a field strength is finite, then $d\varphi = 0$ only when the angle between the vectors $\vec{E}$ and $d\vec{s}$ is 90° (right angle). This means that the field strength vectors and thereby the field lines are perpendicular to the equipotential areas.

The next diagrams show the equipotential areas for a point charge and for a parallel plate assembly. Because of the spherical symmetry of the point charge, the equipotential areas are also spherical. Since the field lines are perpendicular to the conducting surfaces, the conducting surfaces are always equipotential areas [5].



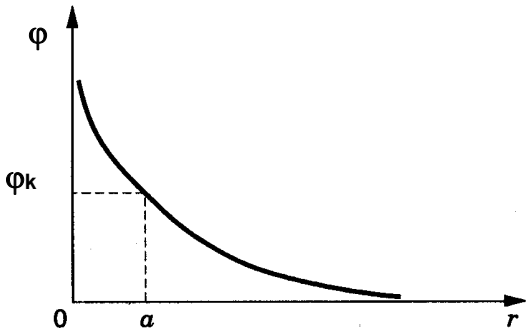
Finally the potential of a positive point charge is calculated (see diagram on the left). We therefore determine that the point P_1 is located in infinity and that the potential there equals to zero. We integrate along a field line from ∞ to the surface in the interval *r*. Then:

$$\vec{E} \cdot d\vec{s} = -|E| \cdot ds = |E| \cdot dr \ ,$$

because both vectors $\vec{E}$ and $d\vec{s}$ have opposite directions, and $d\vec{s}$ = - $d\vec{r}$.

For the electric field strength *E*, we can now insert the causative charge *Q*. Thus:

$$-\varphi(r) = \int_{\infty}^{r} \frac{1}{4\pi \cdot \varepsilon} \cdot \frac{Q}{r^2} dr = \frac{1}{4\pi \cdot \varepsilon} \cdot Q \cdot \int_{\infty}^{r} \frac{dr}{r^2}$$. Thus ensues: $\varphi(r) = \frac{1}{4\pi \cdot \varepsilon} \cdot \frac{Q}{r}$



This potential progression is depicted in the next diagram.

At the location of the point charge an infinitely high potential would occur. This is a further indication that ideal point charges exist in theory but not in reality (compare to 3.3).

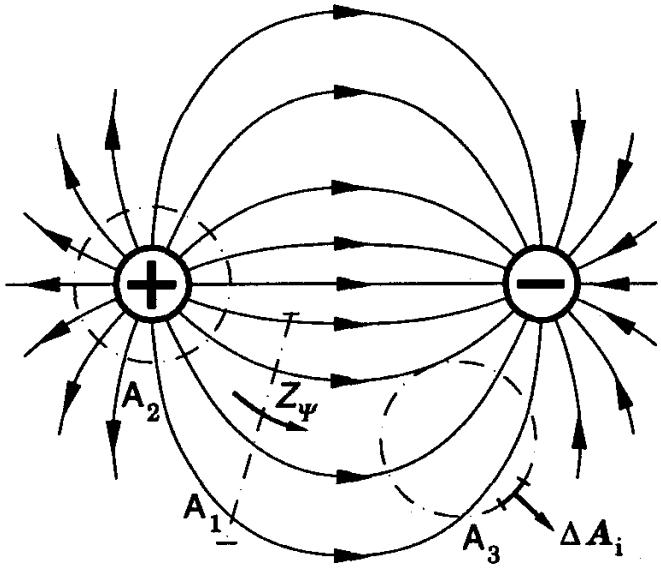
3.5 Excitation of the electrostatic field

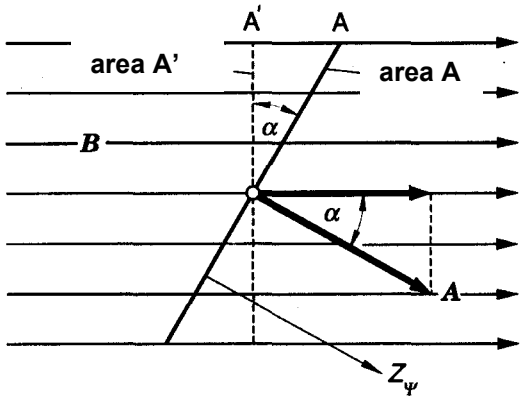
3.5.1 General information about flux

An important term of a vector field is the so-called flux. For the calculation of a flux, the definition of an area is needed. This area can be open or completely surrounding a volume (closed area). An example for an open area is a leaf, an example for a closed area is the surface of a sphere. The flux identifies how many field lines of a vector field pass through this defined area. Usually the flux is considered to be positive when the field lines exit this area.

In the next diagram, different areas are drawn in the field of two point charges. The area A_1 is open, the area A_2 is closed and contains a point charge. The area A_3 is closed, but contains no point charge.

If areas A_2 and A_3 are compared with each other, it is noticeable that on area A_2 all the field lines exit through the exterior surface. The flux of this area would not be equal to zero. In the case of area A_3 the number of field lines entering the area is equal to the number of field lines leaving the area. The flux is therefore zero.





For an open area, the positive orientation of this area must be defined by a normal vector. As shown in the next diagram, this was done with the assistance of the area vector *A*.

Flux can be calculated using simple multiplication if the field is homogenous. If the directions of field vector *A* and field vector *B* are not the same, the reference area *A'* has to be used instead of the real area *A*. The flux results as a scalar product of the vector quantities *B* and *A'*:

$$\Psi = \left|B\right| \cdot A'$$

If the field is not homogenous, the flux has to be calculated through addition of individual area segments:

$$\Psi = \sum_{i=1}^{n} \vec{B}_i \cdot \Delta\vec{A}_i$$

In general, the flux is determined by the area integral:

$$\Psi = \int_A \vec{B} \cdot d\vec{A}$$

This is the general definition of flux. Please note that *B* and *A* are vector quantities.

3.5.2 Electric flux

Several theories have been developed to describe the effect of the electric field. With the so-called action-at-a-distance theory the effect of charges over a distance between the charges are observed. The close-range-effect theory, according to Faraday, assumes that not the charges but rather the space is the carrier of the electric forces. These propagate, starting at the charges, from point to point into space.

Another often-used approach is the definition of electric excitation. In this case, it is assumed that in every field point there is correlating excitation of the electric state. A new vector is allocated to this excitation and identified as “electric excitation”.

Thereby, the electric field strength depends not only on the charge distribution, but also on the medium in which the charges are located.

Example: point charge $$E = \frac{Q}{4 \cdot \pi \cdot \varepsilon \cdot r^2}$$

Therefore, a physical quantity shall be defined, which describes the influence of a charge *Q(x,y,z)* on the surrounding space, without considering the surrounding medium.

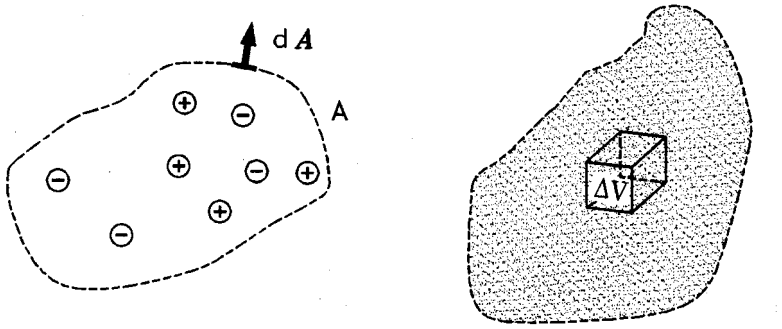
Simplest case: Point charge in any homogenous medium, which contains no other charges in the vicinity.

The influence that the charge has on the surrounding space depends solely on the value of the charge itself. Therefore, the quantities charge and influence are identical:

Electric (displacement) flux $\boxed{\Psi = Q}$ "Psi"

Attention: „Displacement flux is a very misleading description. Here, nothing flows, nothing is being displaced either, especially not a charge. It is more a matter of complete influence over the surrounding space (better: "excitation flux").

The above thought can be further generalized. The complete influence of electric charges on the surrounding space can be interpreted as total flux of electric excitation. This is, in any closed area, equal to the total value of the enclosed charges [5].



a) single charges **b) space charge**

With single charges, the total charge is simply the sum of the single charges:

$$Q = \sum_{i=1}^{n} Q_i$$

If the so-called area charge is defined, the total charge must be calculated using the volume integral as follows:

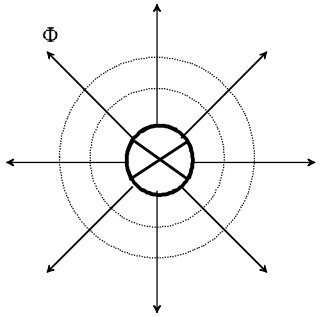
$$Q = \sum \rho_i \Delta V_i = \int_V \rho \cdot dV$$

3.5.3 Electric flux density

The charge's influence spreads with increasing distance onto continually larger (spherical) surfaces. This is comparable to the luminous flux, which extends from a point-like light source.

Electric flux is an integral scalar quantity without indication of a direction. A vectorial quantity which describes the charge influence and its direction on a definite, spatially limited point is also needed.

Simplest case:



Point charges in any homogenous medium with no other charges nearby

Electric displacement flux

Ψ Unit [As]

Electric flux density

$D = \frac{d\Psi}{dA}$ Unit [As/m^2]

Analogy:

Luminous flux	Φ	Unit [lm]
Light intensity	$E = \frac{d\Phi}{dA}$	Unit [lm/m^2 = lx]

A charge does not generally affect the surrounding space evenly in all directions (the luminous flux of a lamp must not distribute evenly in all directions either).

Therefore, the definition of **electric flux density** (analogous to light density):

$$\boxed{D = \frac{d\Psi}{dA}}$$ Condition: Area *dA* is perpendicular to the direction of flow

Thus ensues generally: $\boxed{\vec{D} = \frac{d\Psi}{d\vec{A}}}$ Unit $[D] = \frac{As}{m^2}$

Inversion of this equation results in:

$$\Psi = \oint_A \vec{D} \cdot d\vec{A} \quad \Longrightarrow \quad \boxed{Q = \oint_A \vec{D} \cdot d\vec{A}}$$ „ Gauss‘ law“

$\oiint$ or also $\oint$: The integral extends over a closed surface area – “surface integral”

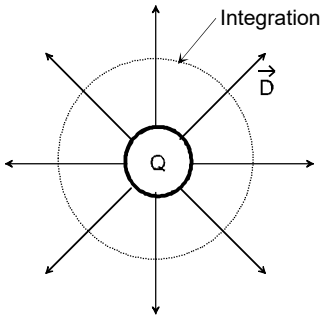
The above mentioned Gauss’ law can be formulated more generally as follows (3.3.4). Thereby the volume V will be enclosed by the surface area *A*:

$$Q = \int_V \rho \cdot dV = \oint_A \vec{D} \cdot d\vec{A}$$

3.5.4 Permittivity

Further example: point charge

Integration of the electric flux density over the surface of a concentric sphere. There *D* has the same value everywhere.



$$Q = \oint \vec{D} \cdot d\vec{A}$$

$$Q = D \cdot A_{Kugel}$$

$$Q = D \cdot 4 \cdot \pi \cdot r^2$$

$$D = \frac{Q}{4 \cdot \pi \cdot r^2} \text{ mit } D = \varepsilon \cdot E \quad \text{folgt} \quad E = \frac{Q}{4 \cdot \pi \cdot \varepsilon \cdot r^2}$$

Thus ensues: $\varepsilon = \frac{D}{E}$ (ε = permittivity)

In this example, $\vec{D}$ and $\vec{E}$ radiate out from the point charge; they have the same direction.

$$\boxed{\vec{D} = \varepsilon \cdot \vec{E}}$$

number value for the vacuum is the **electric field constant:**

$$\varepsilon_0 = \frac{1}{36 \cdot \pi} \cdot 10^{-9} \frac{As}{Vm} = 8{,}854 \cdot 10^{-12} \frac{As}{Vm}$$

This equation can be generally applied.

with $\boxed{\varepsilon = \varepsilon_0 \cdot \varepsilon_r}$.

$\varepsilon_r =$ permittivity number

	air	glass	paxolin	pertinax	oil
ε_r	1,0006	5..10	5..6	4,8	2,2..2,5

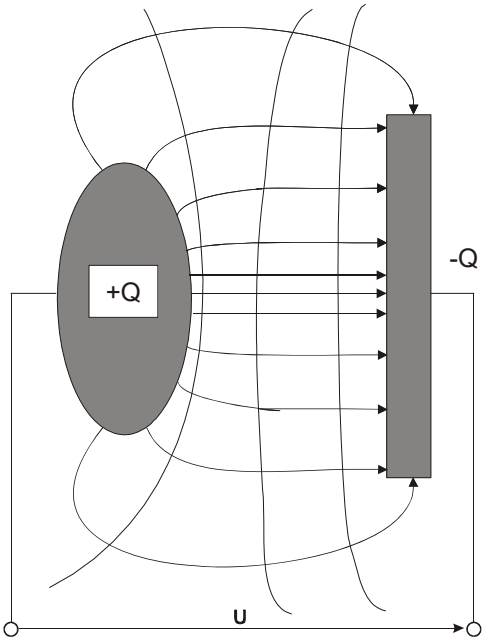
3.6 Capacitance

3.6.1 Definition

In this chapter an important parameter of the electric field is covered. It is the capacitance which indicates how much energy can be stored in the electric field of a system. The capacitance also characterizes an important element of electrical engineering, called the capacitor.

In the technical application insulating materials are always required between conductors or plate assemblies. Although in the simplest case the surrounding air can be chosen as this insulating material, a higher insulating potential is achieved with materials which are identified as dielectrics. The influence of such insulating materials is also a component of this chapter.

If a potential is applied to two electrodes, one of the electrodes will be charged positively and the other negatively. Thereby, proportionality exists between voltage and charge.

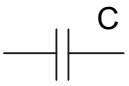


$$Q \sim U$$

$$\boxed{Q = C \cdot U}$$

$$C = \frac{Q}{U}$$ **Capacitance**

$$\text{Unit}\left[C\right] = \frac{As}{V} = F \text{ (Farad)}$$



Circuit symbol

3.6.2 Methods to calculate capacitance

To calculate capacitance there are two principle methods:

- impressed charge *Q*
- impressed voltage *U*.

Method *Q-D-E-U-C*

With this method the test charge +*Q* is applied to the one electrode of the capacitor and the charge –*Q* is applied to the other. This method is always useful when the electric excitation on an equipotential area has the same magnitude everywhere and points in one direction of the chosen coordinate system.

Q → specify test charge

D → symmetry → $\boxed{\vec{D} = \frac{Q}{\vec{A}}}$ ($\left|\vec{D}\right|$ constant on equipotential area *A*)

E → $\vec{E} = \frac{\vec{D}}{\varepsilon}$

U → $U = \int_s \vec{E} \cdot d\vec{s}$ Integration along field line

C → $C = \frac{Q}{U}$

Example: simple plate capacitor, cylinder capacitor, spherical capacitor (3.6.3 – 3.6.5), series connection of capacities (3.6.8).

Method *U-E-D-Q-C*

With this method, a test voltage *U* is impressed upon the capacitor, and the resulting charge is determined. This method is always useful when the electric field strength along a field line has the same magnitude everywhere and points towards one direction of the chosen coordinate system.

U → Specify test voltage

E → symmetry → $\boxed{\vec{E} = -\text{grad}\varphi = -\frac{\text{d}\varphi}{\text{d}\vec{s}}}$

($\left|\vec{E}\right|$ constant along the field line *s*)

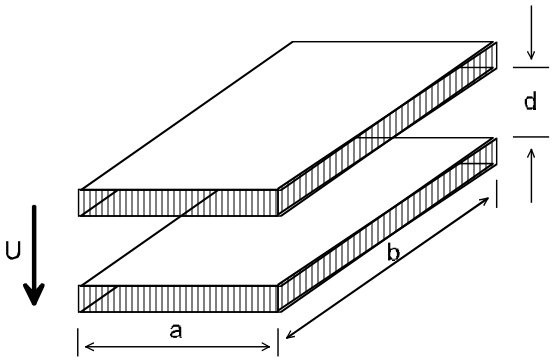
D → $\vec{D} = \varepsilon \cdot \vec{E}$

Q → $Q = \int_A \vec{D} \cdot d\vec{A}$ Integration over equipotential area

C → $C = \frac{Q}{U}$

Example: parallel connection of capacities (3.6.7), vertical layered dielectrics

3.6.3 Plate capacitor



a, b >> d

fringe area field neglected

Method *Q-D-E-U-C*

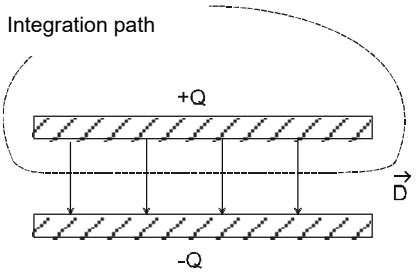
$$Q = \oint \vec{D} \cdot d\vec{A} = D \cdot A \quad \text{mit} \quad A = a \cdot b$$

$$D = \frac{Q}{A}$$

$$E = \frac{D}{\varepsilon} = \frac{Q}{\varepsilon \cdot A}$$

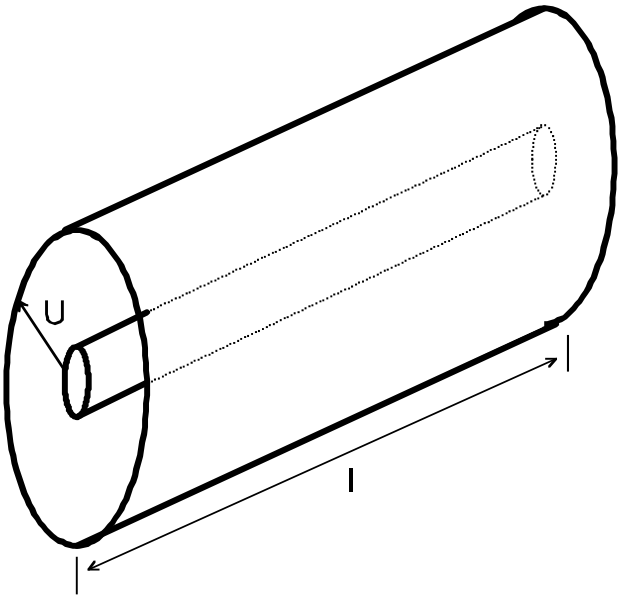
$$U = \int E \cdot ds = E \cdot d = \frac{Q \cdot d}{\varepsilon \cdot A}$$

$$C = \frac{Q}{U} = Q \cdot \frac{\varepsilon \cdot A}{Q \cdot d}$$



$$\boxed{C = \frac{\varepsilon \cdot A}{d}} \qquad \boxed{C = \frac{\varepsilon_0 \cdot \varepsilon_r \cdot A}{d}}$$

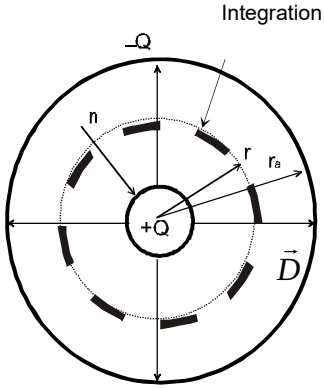
3.6.4 Cylinder capacitor



$l >> r_a - r_i$

margin field neglected

Method *Q-D-E-U-C*



$$Q = \oint \vec{D} \cdot d\vec{A} = D \cdot 2 \cdot \pi \cdot r \cdot l$$

$$D = \frac{Q}{2 \cdot \pi \cdot r \cdot l}$$

$$E = \frac{D}{\varepsilon} = \frac{Q}{2 \cdot \pi \cdot \varepsilon \cdot r \cdot l}$$

$$U = \int_{r_i}^{r_a} E \cdot dr = \frac{Q}{2 \cdot \pi \cdot \varepsilon \cdot l} \cdot \int_{r_i}^{r_a} \frac{1}{r} dr$$

$$U = \frac{Q}{2 \cdot \pi \cdot \varepsilon \cdot l} \cdot \left[\ln(r_a) - \ln(r_i)\right]$$

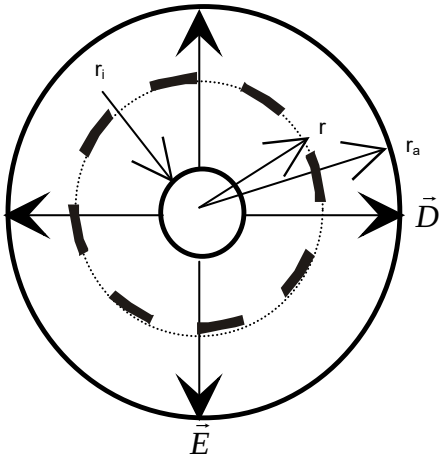
$$U = \frac{Q}{2 \cdot \pi \cdot \varepsilon \cdot l} \cdot \ln\left(\frac{r_a}{r_i}\right)$$

$$C = \frac{Q}{U} = Q \cdot \frac{2 \cdot \pi \cdot \varepsilon \cdot l}{Q \cdot \ln\left(\frac{r_a}{r_i}\right)}$$

$$\boxed{C = \frac{2 \cdot \pi \cdot \varepsilon \cdot l}{\ln\left(\frac{r_a}{r_i}\right)}} \qquad \boxed{C = \frac{2 \cdot \pi \cdot \varepsilon_0 \cdot \varepsilon_r \cdot l}{\ln\left(\frac{r_a}{r_i}\right)}}$$

3.6.5 Spherical capacitor

Method *Q-D-E-U-C*



$$Q = \oint \vec{D} \cdot d\vec{A} = D \cdot 4 \cdot \pi \cdot r^2$$

$$D = \frac{Q}{4 \cdot \pi \cdot r^2}$$

$$E = \frac{D}{\varepsilon} = \frac{Q}{4 \cdot \pi \cdot \varepsilon \cdot r^2}$$

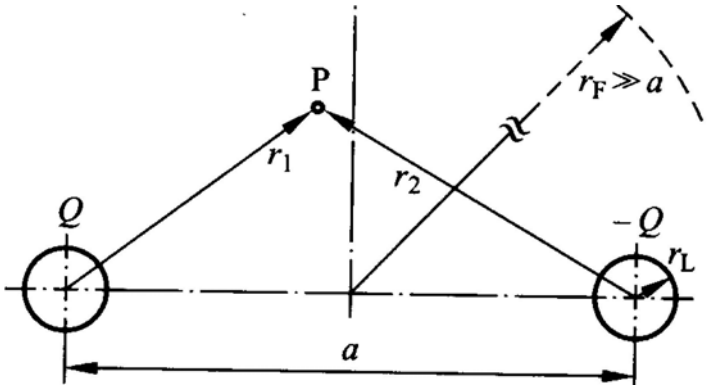
$$U = \int_{r_i}^{r_a} E \cdot dr = \frac{Q}{4 \cdot \pi \cdot \varepsilon} \cdot \int_{r_i}^{r_a} \frac{1}{r^2} dr$$

$$U = \frac{Q}{4 \cdot \pi \cdot \varepsilon} \cdot \left[\frac{1}{r_i} - \frac{1}{r_a} \right]$$

$$C = \frac{Q}{U} = Q \cdot \frac{4 \cdot \pi \cdot \varepsilon}{Q \cdot \left[\frac{1}{r_i} - \frac{1}{r_a} \right]}$$

$$\boxed{C = \frac{4 \cdot \pi \cdot \varepsilon_0 \cdot \varepsilon_r}{\frac{1}{r_i} - \frac{1}{r_a}} = 4 \cdot \pi \cdot \varepsilon_0 \cdot \varepsilon_r \cdot \frac{r_i \cdot r_a}{r_a - r_i}}$$

3.6.6 Capacitance of a two-wire line



Method Q-D-E-U-C

The method of superposition must be applied here additionally. Both conductors are treated independently from each other. In this way the influence ($\vec{D}$, $\vec{E}$) on the surroundings is calculated separately [2].

It is deemed: $r_L << a$

We calculate the voltage of a point P towards a cylinder around both conductors, the point has a radius $r_F >> a$. The cylinder potential is 0.

First step:
only the left conductor is observed (+*Q*)

$$D_1 = \frac{Q}{2\pi \cdot r_1 \cdot l}$$

$$E_1 = \frac{Q}{2\pi \cdot \varepsilon_0 \cdot r_1 \cdot l}$$

$$U_{P1} = \int_{r_1}^{r_F} \frac{Q}{2\pi \cdot \varepsilon_0 \cdot l} \cdot \frac{1}{r} \cdot dr$$

$$U_{P1} = \frac{Q}{2\pi \cdot \varepsilon_0 \cdot l} \cdot \ln\left(\frac{r_F}{r_1}\right)$$

Second step:
only the right conductor is observed (-*Q*)

$$D_2 = \frac{-Q}{2\pi \cdot r_2 \cdot l}$$

$$E_2 = \frac{-Q}{2\pi \cdot \varepsilon_0 \cdot r_2 \cdot l}$$

$$U_{P2} = \int_{r_2}^{r_F} \frac{-Q}{2\pi \cdot \varepsilon_0 \cdot l} \cdot \frac{1}{r} \cdot dr$$

$$U_{P2} = \frac{-Q}{2\pi \cdot \varepsilon_0 \cdot l} \cdot \ln\left(\frac{r_F}{r_2}\right)$$

The voltage U_P is given by the superposition of the two partial charges:

$$U_p = U_{P1} + U_{P2} = \frac{Q}{2\pi \cdot \varepsilon_0 \cdot l} \cdot \left[\ln\left(\frac{r_F}{r_1}\right) + \ln\left(\frac{r_2}{r_F}\right)\right] = \frac{Q}{2\pi \cdot \varepsilon_0 \cdot l} \cdot \ln\left(\frac{r_2}{r_1}\right)$$

If we want to know the voltage between the two conductors, we must calculate the voltage on the surface of both conductors, U_{L1} and U_{L2}. This means we must insert in the above equation: $r_2 \rightarrow a$ and $r_1 \rightarrow r_L$ and vice versa:

Voltage from conductor 1:

$$U_{L1} = \frac{Q}{2\pi \cdot \varepsilon_0 \cdot l} \cdot \ln\left(\frac{a}{r_L}\right)$$

Voltage from conductor 2:

$$U_{L2} = \frac{-Q}{2\pi \cdot \varepsilon_0 \cdot l} \cdot \ln\left(\frac{a}{r_L}\right)$$

Thereby, we can obtain the voltage between both conductors as their potential difference and, consequently, also the capacitance of a two-wire cable:

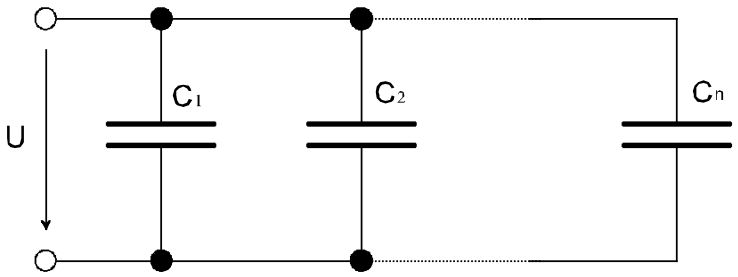
$$U = U_{L1} - U_{L2} = \frac{Q}{\pi \cdot \varepsilon_0 \cdot l} \cdot \ln\left(\frac{a}{r_L}\right) \qquad \text{and} \qquad \boxed{C = \frac{Q}{U} = \frac{\pi \cdot \varepsilon_0 \cdot l}{\ln\left(\frac{a}{r_L}\right)}}$$

3.6.7 Parallel connection of capacitances

As the case of electric resistances, there are two possible basic circuit arrangements when interconnecting multiple capacitors. These are the parallel and series circuits.

In the parallel connection of capacitors, the plates with positive and negative charges are arranged next to each other. The respective positively or negatively charged plates have the same electric potential, the electric voltages on the individual

capacitors are equal. The total charge of the parallel connected capacitors is the sum of the individual charges.



Method *U-E-D-Q-C*

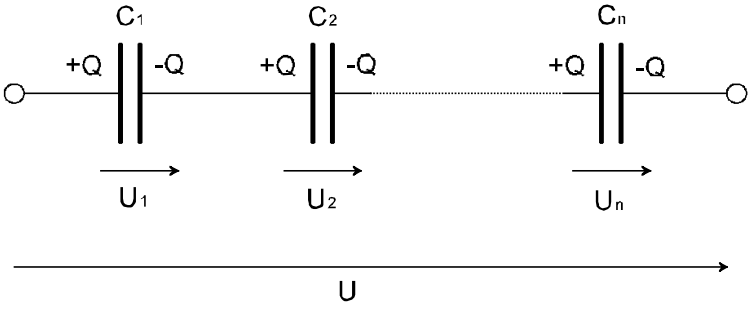
$$Q = Q_1 + Q_2 + ... + Q_n$$
$$Q = C_1 \cdot U + C_2 \cdot U + ... + C_n \cdot U$$
$$Q = (C_1 + C_2 + ... + C_n) \cdot U$$
$$C = \frac{Q}{U} = C_1 + C_2 + ... + C_n$$

$$\boxed{C = \sum_{\nu=1}^{n} C_\nu}$$

We note: **In a parallel connection, the individual capacitances add up to the total capacitance.**

3.6.8 Series connection of capacitances

The capacitors are connected in a row in an uncharged condition. There is no stray current flowing. All capacitors have the same charge. When the voltage source is connected, the left electrode of C_1 gets the charge $+Q$ and the right electrode of C_n gets the charge $-Q$. The charging of the other electrodes occurs through separation of charges.



Method *Q-D-E-U-C*

$$U = U_1 + U_2 + ... + U_n$$
$$U = \frac{Q}{C_1} + \frac{Q}{C_2} + ... + \frac{Q}{C_n}$$

$$\frac{1}{C} = \frac{U}{Q} = \frac{1}{C_1} + \frac{1}{C_2} + ... + \frac{1}{C_n}$$

$$\boxed{\frac{1}{C} = \sum_{\nu=1}^{n} \frac{1}{C_\nu}}$$

We note: **In the series connection, the reciprocals of the individual capacitances add up to the reciprocal of the total capacitance.**

3.6.9 Summary

The following solving strategy can be used to calculate capacitance:

1. Choose an appropriate coordinate system

2. Heed the marginal conditions
   - ideal conducting electrodes ($\vec{E}$ and $\vec{D}$ are perpendicular to the metal)
   - different dielectrics (E_t continuous and D_n continuous)

3. Draw in field line diagram with equipotential areas

4. Consideration of symmetry

   a) If the electric flux density $\vec{D}$ on an equipotential area has the same magnitude everywhere

   ⇨ method ***Q-D-E-U-C***, this is equivalent to a **series circuit** of infinitesimal capacitors

   $$\frac{1}{C} = \int d\frac{1}{C} = \frac{1}{Q}\int dU = \int \frac{ds}{\varepsilon \cdot A}$$

   b) If the electric field strength $\vec{E}$ has the same magnitude everywhere along a field line

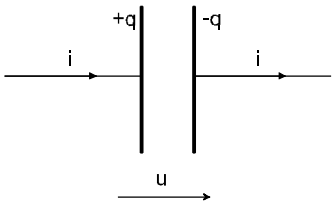
   ⇨ method ***U-E-D-Q-C***, this is equivalent to a **parallel circuit** of infinitesimal capacitors

   $$C = \int dC = \frac{1}{U}\int dQ = \int \frac{\varepsilon \cdot dA}{s}$$

dielectric	**type and character**
synthetic materials	▪ coiled, metal vaporized synthetic foils ▪ dielectric thickness d ≈ 1 … 3 μm ▪ relative permittivity $\varepsilon_r \approx 2 \ldots 4$ ▪ typical C-values 10 pF … 10 μF ▪ standard capacitor (SMD or wired)
ceramic	▪ ceramic multi-layered capacitors: alternating layers of dielectric ceramics (d ≈ 1…3 μm) and metallic electrodes in parallel series ▪ relative permittivity - paraelectric ceramic: $\varepsilon_r \approx 10 \ldots > 100$ - ferroelectric ceramic: $\varepsilon_r \approx 1000 \ldots > 10000$ ▪ high value standard capacitor (SMD) ▪ C-values: 1 pF … 3,3 µF
electrochemically made oxide	▪ dielectric = oxideskin (d ≈ 0,05 … 0,5 μm) of an anodic oxidized metal (aluminium foil or tantalum sinterbody) ▪ description: electrolytic capacitor ▪ relative permittivity $\varepsilon_r \approx 8$ (Al_2O_3), $\varepsilon_r \approx 27$ (Ta_2O_5) ▪ advantage: high C-values (> 10 mF) ▪ disadvantage: high leak current, only NF area
electrochemical double layer	▪ dielectric = adsorbed H_2O- molecule layer (d ≈ 0,1nm !) on precious metal foil (or akin) ▪ relative permittivity $\varepsilon_r \approx 13$ (aligned H_2O-molecules) ▪ advantage: extremely high C-values (> 1 F) ▪ disadvantage: very low nominal voltage, very high leak current, only NF-range

3.7 Energy and forces in electrical fields

3.7.1 Capacitor energy



A capacitor is charged. Voltage and charge increase according to random time functions *u(t)*, or *q(t)*. At the point in time $t = t_1$ the charging is finished. Thus ensues:

$$i(t \geq t_1) = 0 \qquad u(t \geq t_1) = U \qquad q(t \geq t_1) = Q$$

Annotation: Time-dependent quantities are identified with small letters (*u, i, q*).

The following conditions apply:

$$p = u \cdot i$$

$$u = \frac{q}{C};\ i = \frac{dq}{dt}$$

Thus results the energy which is stored in the capacitor for:

$$W = \int_0^{t_1} p \cdot dt$$

$$W = \int_0^{t_1} u \cdot i \cdot dt$$

$$W = \int_0^{t_1} \frac{q}{C} \cdot \frac{dq}{dt} \cdot dt$$

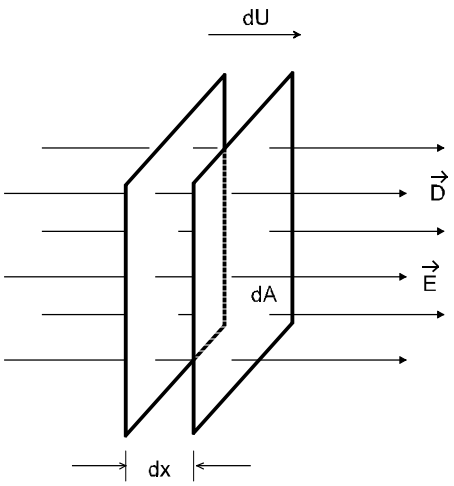
$$W = \int_0^{Q} \frac{1}{C} \cdot q \cdot dq$$

$$W = \frac{1}{2 \cdot C} \cdot q^2 \Big|_{q=0}^{Q}$$

$$\boxed{W = \frac{1}{2} \cdot \frac{Q^2}{C} = \frac{1}{2} \cdot Q \cdot U = \frac{1}{2} \cdot C \cdot U^2}$$

3.7.2 Energy density

In almost all cases, the field distribution is not homogenous. That is, why it is necessary to work with the term ‘energy density’ instead of the ‘total energy’. To deduce the energy density, (virtually) a small plate capacitor is brought into the inhomogeneous field. The dimensions of this plate capacitor are chosen in a way, so that within the capacitor a homogenous field can be assumed. Thus, it is deemed: an infinite small section of this inhomogeneous field can be considered homogenous.



$$dW = \frac{1}{2} \cdot dC \cdot (dU)^2 \quad \text{with } \mathrm{d}C = \frac{\varepsilon \cdot dA}{dx} \text{ and } \mathrm{d}U = E \cdot dx$$

$$dW = \frac{1}{2} \cdot \frac{\varepsilon \cdot dA}{dx} \cdot (E \cdot dx)^2$$

$$dW = \frac{1}{2} \cdot \varepsilon \cdot E^2 \cdot dA \cdot dx$$

$$\boxed{\frac{dW}{dV} = \frac{1}{2} \cdot \varepsilon \cdot E^2 = \frac{1}{2} \cdot D \cdot E = \frac{D^2}{2 \cdot \varepsilon}}$$

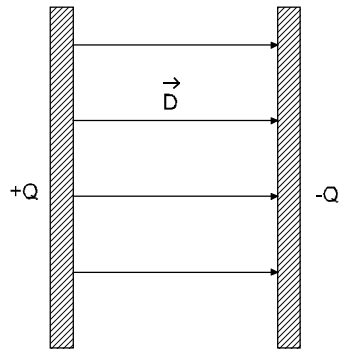
3.7.3 Forces between two electrodes

Calculation method: Only one electrode is examined and its field is determined.

The other electrode is located in this field, so that their charges are affected by forces (compare to 3.2.2)

Example: plate capacitor:

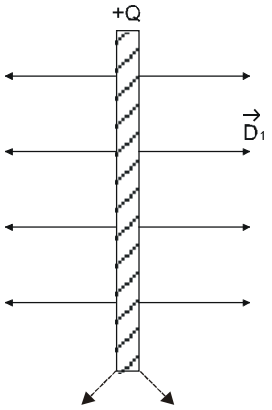
original assembly (two plates)



$$D = \frac{Q}{A}$$

$$E = \frac{Q}{\varepsilon \cdot A}$$

field examination of only the positive plate (without 2nd plate!)



fringe area fields neglected

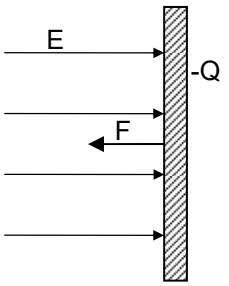
The positive charge spreads to both sides of the 1st plate.

$$Q = \oint D_1 \cdot dA = D_1 \cdot 2 \cdot A$$

$$D_1 = \frac{Q}{2 \cdot A} = \frac{1}{2} \cdot D$$

$$E_1 = \frac{Q}{2 \cdot \varepsilon \cdot A} = \frac{1}{2} \cdot E$$

Therefore *D* and *E* are also reduced by a factor of 2.



The force on the negative (2nd) plate ensues:

$$\vec{F} = -Q \cdot \vec{E}_1 = -\frac{1}{2} \cdot Q \cdot \vec{E}$$

Attraction force according to magnitude:

$$F = \frac{1}{2} \cdot Q \cdot E = \frac{1}{2} \cdot D \cdot A \cdot E = \frac{1}{2} \cdot \frac{Q^2}{\varepsilon \cdot A}$$

Pressure on the capacitor plates:

$$\frac{F}{A} = p = \frac{1}{2} \cdot D \cdot E$$

Forces in an electric field can be calculated with electrostatic field energy, thereby using the principle of virtual displacement. This principle is shown here, again using the example of a plate capacitor.

According to the principle of virtual displacement, the distance between the plates is changed by a small amount Δz. The change of the field energy is then calculated. Thereby, the force effect is proportional to the change of this force energy.

Because of their opposing charges, the two plates attract each other with the force *F*. If the plates are pulled apart by the amount Δz, then the mechanical work:

$$\Delta W = |F| \cdot \Delta z$$

must be expended. According to the law of conservation of energy, the energy of the field increases by the same amount. Before the change in distance, the field energy was equal to:

$$W = \frac{1}{2} \cdot \frac{1}{C} \cdot Q^2$$

Through the displacement of the two plates, the field energy changes by the amount:

$$\Delta W = \frac{1}{2} \cdot Q^2 \cdot \Delta\left(\frac{1}{C}\right)$$

Since the mechanical work in this process has to be equal to the change of the electric field energy, it is deemed:

$$|F| = \frac{1}{2} \cdot Q^2 \cdot \frac{\Delta\left(\frac{1}{C}\right)}{\Delta z}$$

With $\frac{1}{C} = \frac{d}{\varepsilon \cdot A}$ and after replacing *d* by $d+\Delta z$ we obtain:

$$\frac{1}{C} + \Delta\left(\frac{1}{C}\right) = \frac{d}{\varepsilon \cdot A} + \frac{\Delta z}{\varepsilon \cdot A} \qquad \Delta\left(\frac{1}{C}\right) = \frac{\Delta z}{\varepsilon \cdot A}$$

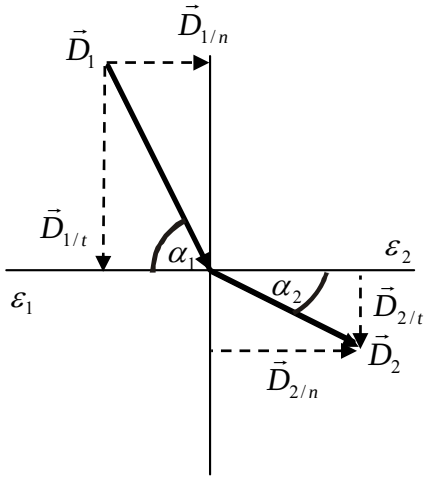
If we enter this in the force equation, as the final result for the force between the plates of a capacitor, we obtain:

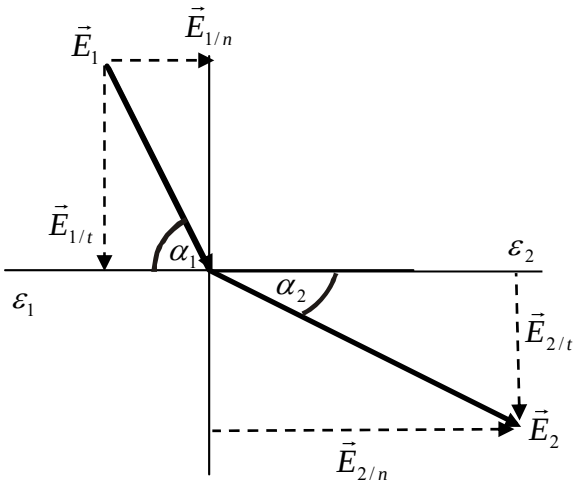
$$|F| = \frac{1}{2} \cdot \frac{Q^2}{\varepsilon \cdot A}$$

3.8 Boundary conditions

The refraction behaviour on boundaries of different mediums is known from optics. In the following, we will examine the boundary between two different materials that have different permittivity values. Furthermore, we assume that the field lines, as depicted in the next picture, are not perpendicular to the boundary.

Based on the different permittivity the magnitude and direction of the field strength in both materials will not be equal. The conditions of irrotationality of the electric field and Gauss' law must be fulfilled on the boundary.





$$\varepsilon_1 \neq \varepsilon_2 \qquad hier: \varepsilon_1 > \varepsilon_2$$

We now take out an infinite small volume element (disc) directly on the boundary between both materials:

Irrotationality (vortex free):

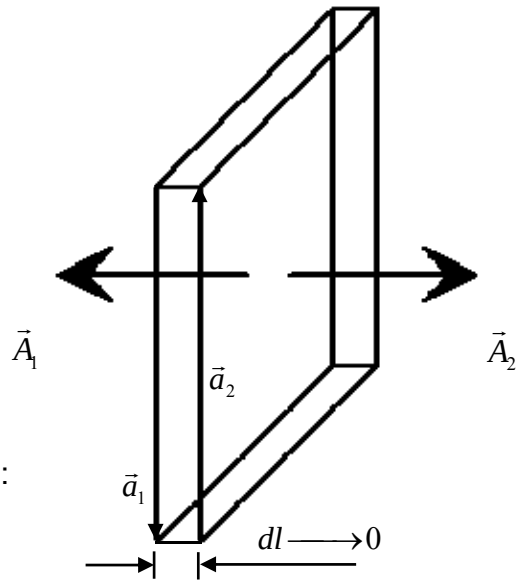
$$\oint_s \vec{E} \cdot d\vec{s} = 0$$

Gauss' law:

$$\oint_A \vec{D} \cdot d\vec{A} = Q$$

Material equation:

$$\vec{D} = \varepsilon \cdot \vec{E}$$



$$|\vec{a}_1| = |\vec{a}_2| = a$$

$$|\vec{A}_1| = |\vec{A}_2| = A$$

A, a: both infinite small

$$\oint_s \vec{E} \cdot d\vec{s} = \vec{E}_1 \cdot \vec{a}_1 + \vec{E}_2 \cdot \vec{a}_2 = E_{1/t} \cdot a - E_{2/t} \cdot a = 0$$

The tangential component of the electric field strength is constant

$$\boxed{E_{1/t} = E_{2/t}}$$

$$\oint_A \vec{D} \cdot d\vec{A} = \vec{D}_1 \cdot \vec{A}_1 + \vec{D}_2 \cdot \vec{A}_2 = -D_{1/n} \cdot A + D_{2/n} \cdot A = 0$$

The normal component of the electric flux density is constant

$$\boxed{D_{1/n} = D_{2/n}}$$

From the combination of both conditions and with $D_{i/n} = \varepsilon_0 \cdot \varepsilon_{ri} \cdot E_{i/n}$ it follows:

$$\boxed{\frac{\tan \alpha_1}{\tan \alpha_2} = \frac{\varepsilon_1}{\varepsilon_2}}$$

This relationship exists only when no electric current flows, this means no electric charges are concentrated on the boundary area (electrostatic field).

When an electric flow field (Ch. 4) and an electrostatic field exist simultaneously, there are electric charges on the boundary. It is deemed:

$$J_{1/n} = J_{2/n}$$

$$\kappa_1 \cdot E_{1/n} = \kappa_2 \cdot E_{2/n}$$

$$\frac{\kappa_1}{\varepsilon_1} \cdot D_{1/n} = \frac{\kappa_2}{\varepsilon_2} \cdot D_{2/n} \quad \rightarrow \qquad \underline{D_{1/n} \neq D_{2/n}}$$

3.9 Matter in the electric field

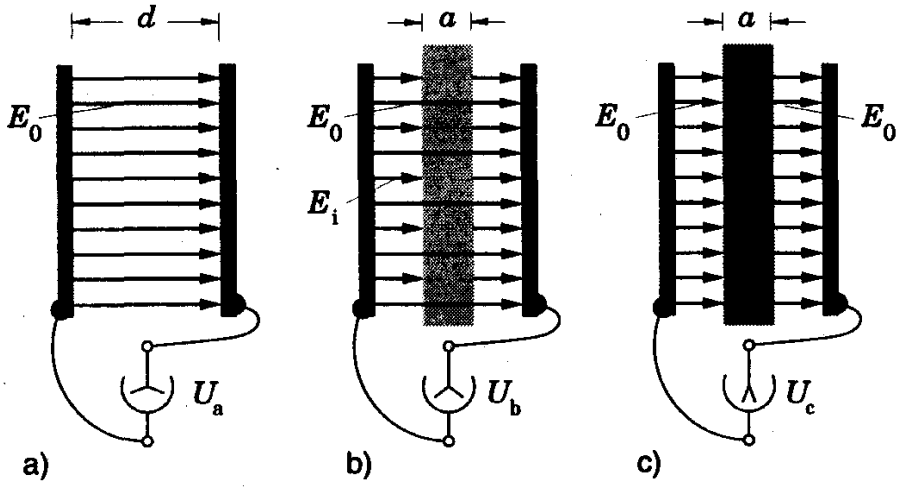
3.9.1 Field strength in insulating materials

The plates of a capacitor must be separated from each other by an insulating layer. The simplest case is air, but in many cases other insulating materials are also used. The question is which influence these materials have on the field of a capacitor. To examine this, the following experiment is conducted.

A plate capacitor is loaded with a charge, so that between the two plates the voltage U_a is set. This charge is not going to be changed throughout the following steps of the experiment. In the next step, an insulating plate with the thickness *a* is inserted between the plates of the capacitor without touching them. It is determined that the voltage changes. The new value is identified as U_b. In a further step, the insulating plate is replaced by a metallised plate. The voltage is measured again. Now, the value becomes U_c. When the voltage values are compared, it is noticed that the voltage is reduced with each step. Therefore it is deemed [2]:

$$U_a > U_b > U_c$$

Since the charges on the plates remained equal in all three cases, the field strength in the matter-free space must have remained constant. The value of this field strength amounts to: $E_0 = \frac{\sigma}{\varepsilon_0}$. σ is the area charge density on the capacitor plates (3.3.3), it is identical to the electric flux density *D*.



In the cases a) and c), we can state the voltage immediately, because no field is present in the metallised plate.

$$U_a = E_0 \cdot d \qquad U_c = E_0 \cdot (d - a)$$

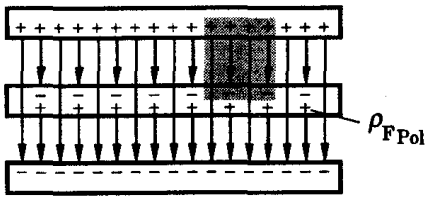
In case b), where the inserted plate is made from dielectric material, a further voltage portion must be added in comparison to case c), because the voltage was higher in case b). We can express the voltage U_b as follows:

$$U_b = E_0 \cdot (d - a) + E_i \cdot a$$

Based on the sequence of the voltage values, the field strength in the insulating material E_i is smaller than the field strength in the free space E_0. With a given charge, the field strength in the matter-filled space is smaller than in the vacuum. Since the insulators can be permeated by an electric field, they are identified as dielectrics.

In the dielectrics a field strength is present which is smaller than the field strength of the free space. A charge separation must have occurred. It is, however, noticeably smaller than the charge separation in the metal plate. The charges in dielectrics are identified as polarization charges.

When Gauss' law is applied to the dielectric, the resulting excitation within the dielectric is the charge density of the capacitor plates minus the charge density of the polarization charges [2]:



$$|D_i| = \varepsilon_0 \cdot |E_i| = \sigma - \sigma_i$$

We allocate the vector P to the density of the polarization charge σ_i. For the entire excitation in the dielectric, we can therefore write:

$$D = D_i + P$$

This equation states that the entire excitation *D*, which is emitted from the sources on the plates, is split within the dielectric into the inner excitation D_i and a counter excitation or polarization *P*.

These considerations are made for a plate capacitor, but they have general validity. In the simplest case, the polarization *P* is proportional to the electric field which is prevalent within the dielectric. The definition of another constant, called susceptibility χ, is expressed as follows:

$$P = \chi \varepsilon_0 E_i$$

Through implementation of the permittivity number (formerly: relative dielectric constant) ε_r the usage of the polarization vector *P* can be avoided: It is then deemed:

$$D = \varepsilon E = \varepsilon_r \varepsilon_0 E$$

$$\varepsilon_r = 1 + \chi$$

material/substance	ε_r	material	ε_r
air	1,00059	quartz	3,8...5
petroleum	2,0	glass	5...7
Polyethylene	2,3	ceramic	9,5...100
polystyrene	2,6	diamond	16,5
rubber	2,5...3,5	nitrobenzene	36,0
amber	2,8	distilled water	81,0

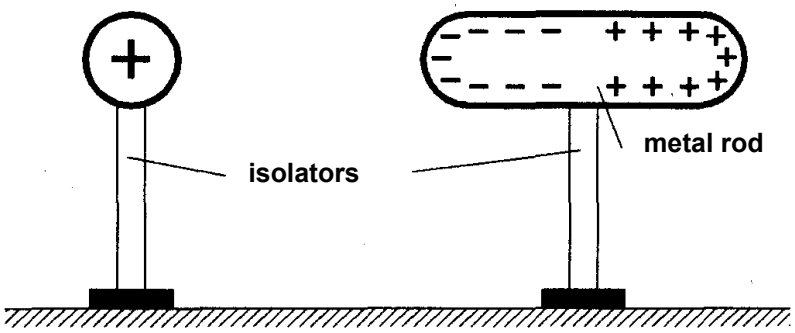
If a capacitor is completely filled with dielectric, its capacitance rises by a factor of ε_r in comparison with the value of vacuum. Thereby the field strength decreases by the same charge uniformly on the plates. To build a capacitor with small dimensions but much capacitance, a dielectric with a large permittivity number must be used. This insulating material must also ensure a high dielectric strength so the spaces and surfaces of the plates can be kept small.

It is important to mention that the permittivity number is not always independent of the electric field strength for all materials.

3.9.2 Electrostatic induction

If a metal rod is, as shown in the next figure, positioned in a field created by a positively charged sphere, a separation of charges occurs in the metal rod. The metal rod exhibits charges on its ends, namely a positive charge on the end pointing away from the sphere and a negative charge on the end pointing to the sphere. These charges are identified as induction charges. The presence of these charges can be verified with a test sphere, with which charges are extracted by touching the ends of the rods and then examining their polarity [2].

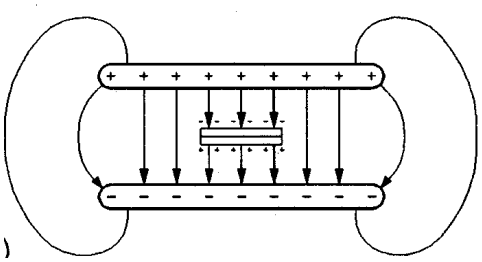
Considering the attraction of negative charges and the repulsion of positive charges, the separation of charges is easily explained.

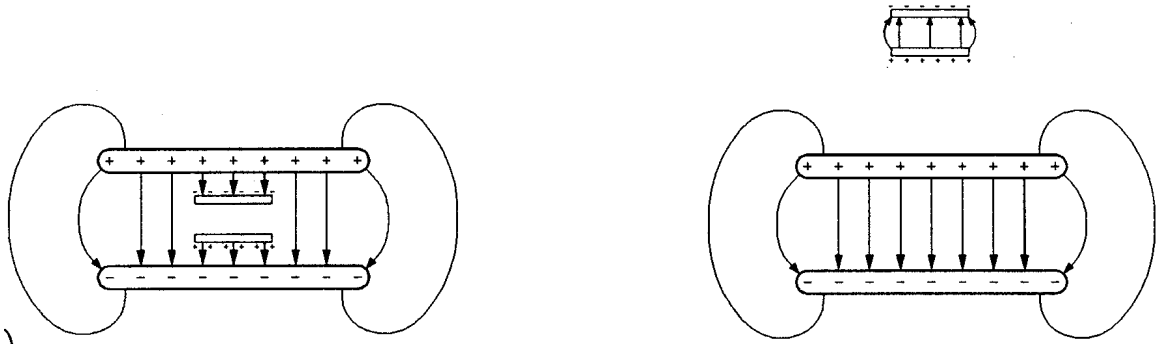


Induction also explains why an uncharged metallised sphere is attracted by a charged rod. The attraction of the inducted charge to the side pointing towards the rod is larger than the repulsion to the other charges. The negative charges in the figure are closer to the sphere and are therefore in an area with higher field strength.

The magnitude of the inducted charge can be determined by separating the two differently charged parts of the rod while within the field. Then each part is examined separately outside the field. Instead of a rod it is better to use two equal metal discs which can be held with insulating handles and are brought into a homogenous electric field (plate capacitor). If the two-disc assembly is thin enough, then the homogenous field is not perceptibly changed as long as the discs are located together and perpendicular to the field.

The two plates are pulled apart and then moved out of the field. The induced charges which were created in the homogenous field of the plate capacitor can now be verified and measured [2].





In a further experiment, the disc area *A* and the electric field strength *E* are changed. This experiment results in the following connection with the induced charge:

$$Q_i = \varepsilon_0 \cdot A \cdot |E|$$

In the above equation, a homogenous electric field was assumed. This equation can be used for an inhomogeneous field as well, if the selected area is very small. For the charge on this small area, it is deemed:

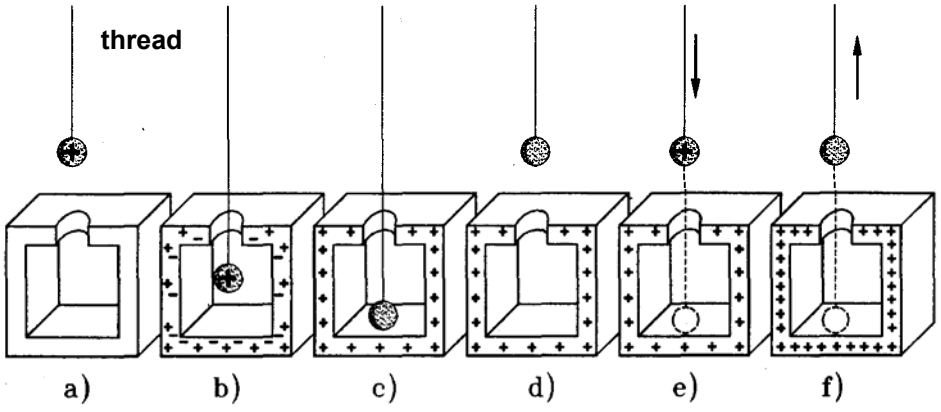
$$\Delta Q_i = \varepsilon_0 \cdot \Delta A \cdot |E|$$

If both sides of this equation are divided by the area segment ΔA, an equation for the area charge density, or the charge per area unit of the induced charges, results (3.3.3):

$$\sigma_i = \varepsilon_0 \cdot |E|$$

3.9.3 Faraday cup

The next experiment delivers further findings and also explains the principles of the induction machine. We will use an uncharged metal box with a small opening in the top which is set on an insulating pedestal [1].



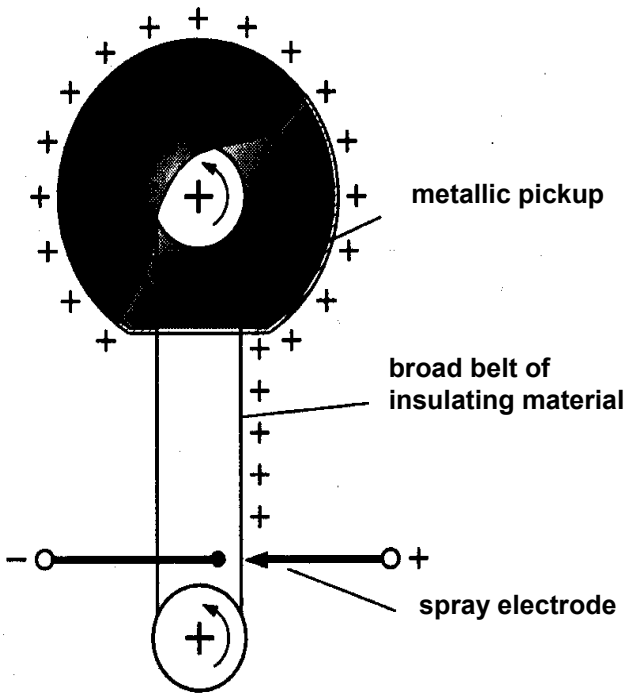
In (a), a positively charged sphere is lowered into the box without touching the wall of the box. In the wall of the box, a charge separation occurs (b). Negative charges gather on the inside, positive charges on the outside. Now the electric field outside of the box is examined. It is determined that it is independent of the sphere's position in the box.

In (c), the sphere touches the box. The charge of the sphere passes over to wall of the box. The box now is positively charged. The sphere has delivered its charge and can be lifted out (d). Outside of the box, this process has no influence on the electric field. We can therefore conclude: The total magnitude of the induced charges on the inside and the outside of the box is, in terms of quantity, equivalent to the magnitude of the sphere's charge.

The result can be stated as follows: By touching the box, the sphere delivers its complete charge. The charge migrates to the outside because no field can exist in the charge-free inside of a conducting hollow body.

In (e) and (f), the immersion of the sphere is repeated several times and thereby further charges are transferred to the box.

This last procedure is continually performed by an electrostatic generator. It is also known as a Van de Graff belt generator. With the help of an insulating belt, on which charges are applied with a spray electrode, the charges are transported into the hollow metallic sphere. Spray electrodes are understood as metal points. The field strength on conducting points can be so high that the insulating potential of air is exceeded. In this case, the charge carriers emitted from the metal points are picked up by the belt of insulating material. Such a generator can create voltages in excess of 10 MV [1].



4. Current field (field of the electric flow)

We will now focus on a field which is caused by the flow of electric currents. In other words, electric charges are moving. The field is stationary, which means time-invariant.

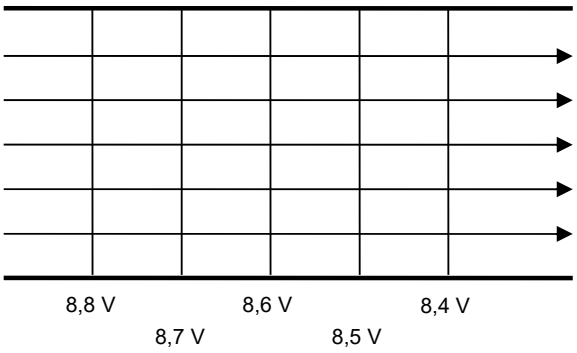
4.1 Electric current density and field strengths

4.1.1 Current lines and equipotential surfaces

We observe current lines and equipotential surfaces in a homogenous wire with constant width or diameter.

Current lines are possible trajectories, through which the electric charges can travel while under the influence of an external voltage gradient. The distance of the current lines presents the current density.

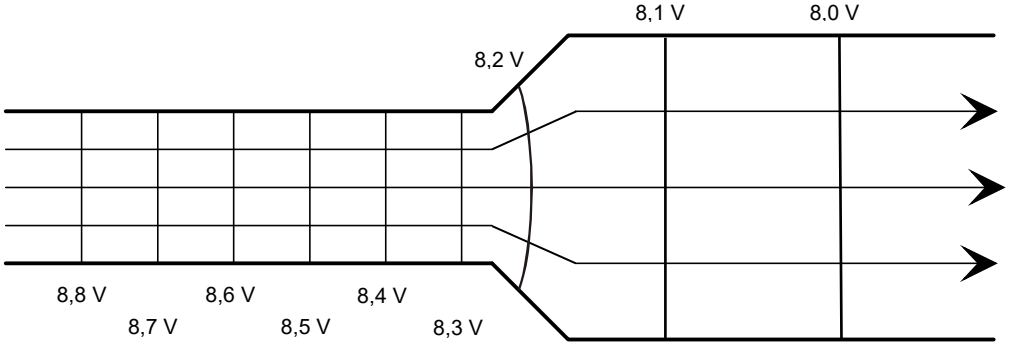
Equipotential surface is an area with equal electric potential, this means no voltage at all exists between points on the same equipotential surface.



In the homogenous field the electric current is evenly distributed throughout the cross section. The current lines are therefore parallel (current lines cannot overlap) and have the same density everywhere.

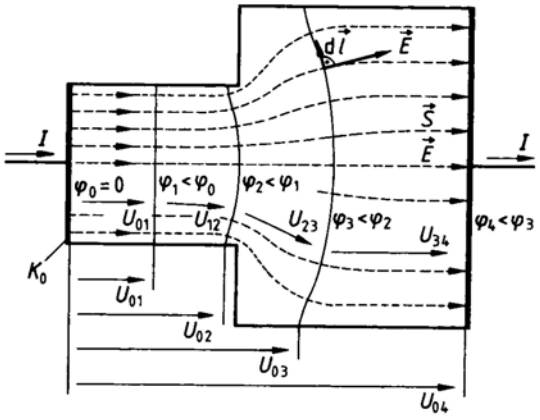
The voltage decreases evenly along the length of the wire. Therefore, the equipotential surfaces are parallel and are the same distance apart from each other (equidistant).

If the width of the wire is changed, the distance of the current lines from each other in the homogenous area must change. Thereby, the equipotential surfaces aren't equidistant anymore.

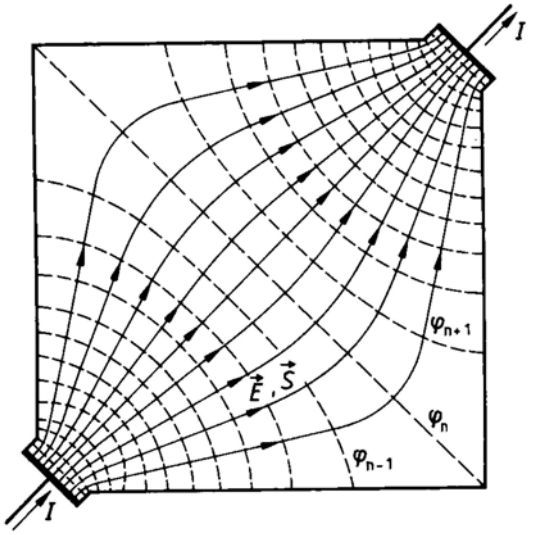


Two further examples show this connection in the quasi-plane (two dimensional) flow field [1].

Wire with change in cross section:



Square plate contacted on two corners:



Annotation: In German technical textbooks the electric density is often represented by the symbol *S* instead of *J*.

Electric charges flow along the shortest path from one equipotential surface to the next. Therefore the current lines are perpendicular to the equipotential surface.

Analogy: Water also flows along the lines of greatest slope (=current lines) from one contour line (=equipotential surface) to the next.

Conclusion: Current lines and equipotential surfaces are perpendicular to each other.

4.1.2 Electric current density

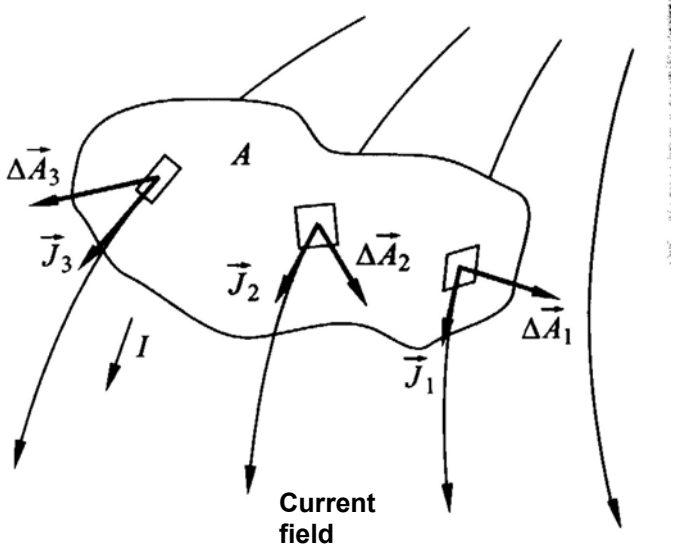
Current density = current per unit area

In a homogenous wire with a constant width the correlation is simple:

$$\boxed{J = \frac{I}{A}}$$

With locally depending current density (inhomogeneous field), such a small observed area must be chosen that the current density is constant over the entire area. This is successful when the area is made infinite small:

$$J \approx \frac{\Delta I}{\Delta A} \qquad \rightarrow \qquad \boxed{J = \frac{dI}{dA}}$$



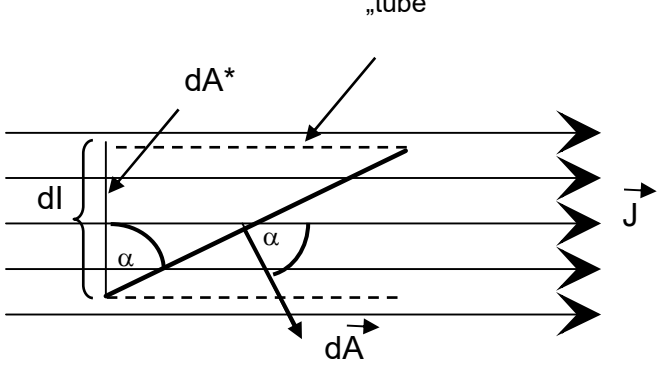
Current field

The requirement in all cases is:
Each area ΔA or dA must be perpendicular to the current direction, this means $\vec{J}$ and $d\vec{A}$ are supposed to have the same direction [1].

Unit [J] = A/m^2

The reverse is also deemed: $dI = J \cdot dA$ and $\boxed{I = \int_A J \cdot dA}$

If, however $\vec{J}$ and $d\vec{A}$ don't show the same direction:



Infinite small section of an inhomogeneous current field

The area vector $d\vec{A}$ is perpendicular to the area. With a closed mantle area, the area vector $d\vec{A}$ always points outward.

Therefore the current in the depicted "tube" results in:

$dI \neq J \cdot dA$ but instead

$$dI = J \cdot dA^*$$

$$dI = J \cdot dA \cdot \cos\alpha$$

$$dI = \vec{J} \cdot d\vec{A}$$

$$I = \int_A dI$$

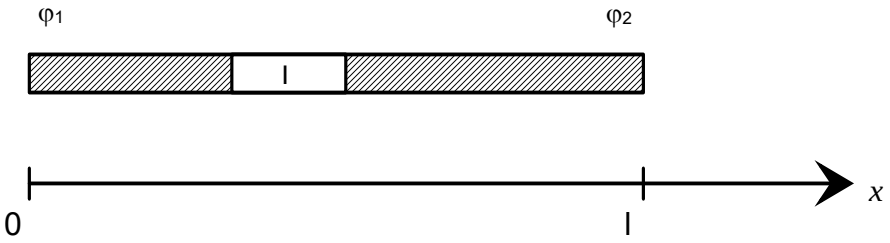
$$\boxed{I = \int_A \vec{J} \cdot d\vec{A}}$$

4.1.3 Electric field strength

The electric field strength indicates how strong the potential decreases depending on the location:

electric field strength = potential decrease

Simplest case: Homogenous wire with constant width:



The potential decreases linearly along the length *l* of the wire. It is deemed:

$$E = \frac{U}{l} = \frac{\varphi_1 - \varphi_2}{l} = -\frac{\varphi_2 - \varphi_1}{l} = -\frac{\Delta\varphi}{l}$$

When the wire resistance is not spread evenly over the complete wire length, for example because the conductivity changes or the width changes, it is deemed:

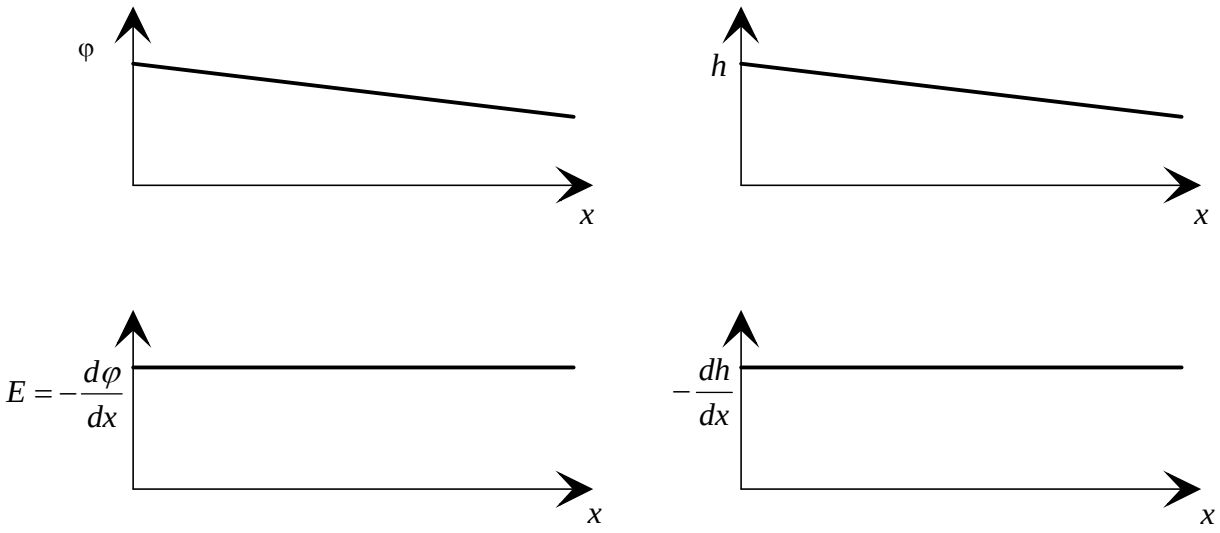
$$E = -\frac{d\varphi}{dx}$$ Potential <u>decrease</u> per path segment

Unit [*E*] = V/m

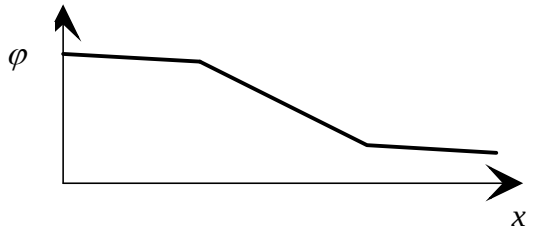
We can also clearly establish the connection between electric field strength and potential with a simple analogy, the gradient (or slope) of a street.

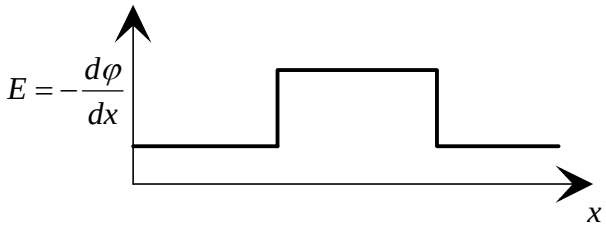
<u>Analogy:</u> Electric field strength in a wire ↔ slope of a street

1. Homogenous wire constant width — street with constant downgrade

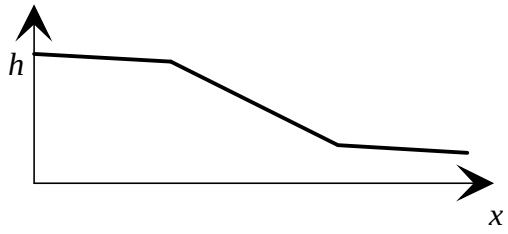


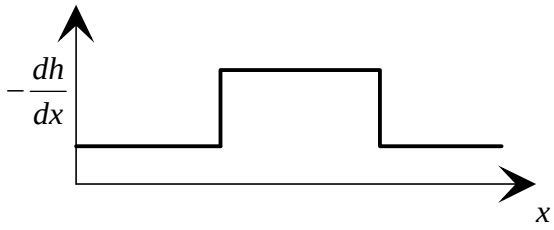
2. Inhomogeneous wire





street with varying downgrade





We try to formulate the connection more generally, first for a two dimensional observation:

The downgrade at one point in the area can be formulated as:→ $-\frac{dh}{dl}$

This statement is not definite, because the value of $-dh/dl$ depends on the direction of *dl*. The term “downgrade” here means the downgrade in direction of the line of greatest slope. The downgrade does not only have an <u>amount</u> but also a <u>direction</u>, this means it is a vector!

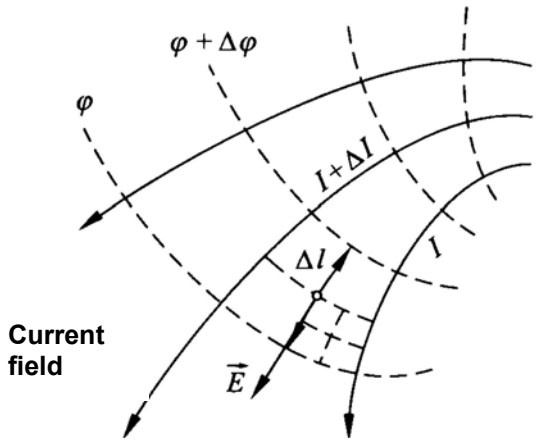
For this there is a mathematical tool: Gradient

grad *h* is a vector

with $\left|grad\ h\right| = \frac{dh}{dl}$ and $\frac{grad\ h}{\left|grad\ h\right|} = \frac{d\vec{l}}{dl}$

$d\vec{l}$ is the path segment in the direction of the steepest incline

The value of **grad *h*** indicates the value of the slope, the direction of **grad *h*** indicates the direction of the slope.



The decline has the same value as the incline, but in the opposite direction [19:

decline = - grad *h*

The same consideration can also be applied to the potential slope:

$$\boxed{\vec{E} = -grad\varphi}$$

This equation for $\vec{E}$ can also be applied unmodified to three-dimensional fields.

To ascertain the direction of grad φ (= direction of the steepest incline), the potential change in direction of the three spatial coordinates (here the Cartesian coordinates) must be determined:

Definition: $$\boxed{grad\varphi = \begin{pmatrix} \frac{\partial\varphi}{\partial x} \\ \frac{\partial\varphi}{\partial y} \\ \frac{\partial\varphi}{\partial z} \end{pmatrix}}$$

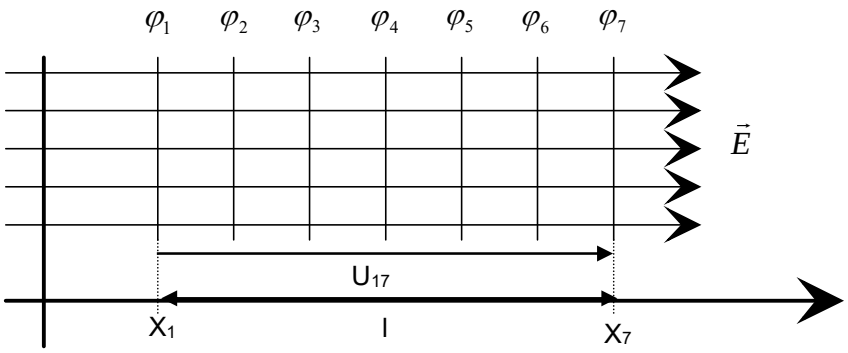
We can thereby formulate the connection between voltage and electric field strength as follows.

1. For a homogenous electric field:

$$E = -\frac{d\varphi}{dx} = -\frac{\varphi_7 - \varphi_1}{x_7 - x_1}$$

$$E = \frac{\varphi_1 - \varphi_7}{x_7 - x_1} = \frac{U_{17}}{l}$$

$$\rightarrow \boxed{U_{17} = E \cdot l}$$

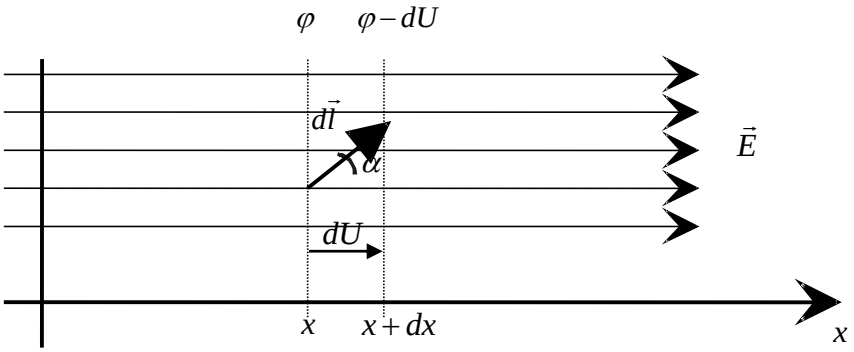


2. For an infinite small area of an inhomogeneous electric field:

$$E = -\frac{d\varphi}{dx} = -\frac{\varphi - dU - \varphi}{dx}$$

$$E = \frac{dU}{dx}$$

$$\rightarrow \boxed{dU = E \cdot dx}$$



When the field strength vector $\vec{E}$ and the path segment vector $d\vec{l}$ do not have the same direction, it is deemed:

$$dU = E \cdot dl \cdot \cos\alpha = \vec{E} \cdot d\vec{l}$$

The integration path is arbitrary, it simply begins at P_1 and ends at P_2.

$$\boxed{U = \int_{P_1}^{P_2} \vec{E} \cdot d\vec{l}}$$

4.2 Ohm's law in differential form

During the study of networks, we learned about Ohm's law. By using the physical quantities electric current density and electric field strength, it can be generalized, which means it is formulated in a differential form.

4.2.1 Generalized Kirchhoff's current law

We know: At a node the sum of all currents must be 0. If we replace the currents by the current densities which have been integrated over the respective areas, then it is deemed:

$$\boxed{\oint_A \vec{J} \cdot d\vec{A} = 0}$$

To recreate the node the integration has to be done over a closed surface.

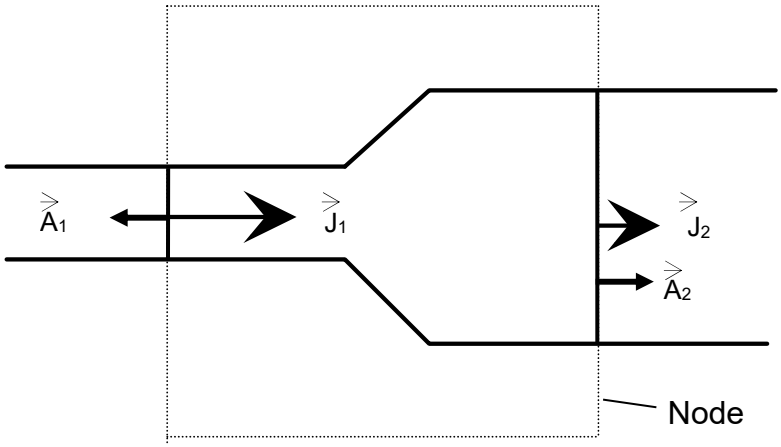
Example:

$$\vec{J}_1 \cdot \vec{A}_1 + \vec{J}_2 \cdot \vec{A}_2 = 0$$

$$-J_1 \cdot A_1 + J_2 \cdot A_2 = 0$$

$$J_1 \cdot A_1 = J_2 \cdot A_2$$

$$\frac{J_2}{J_1} = \frac{A_1}{A_2}$$

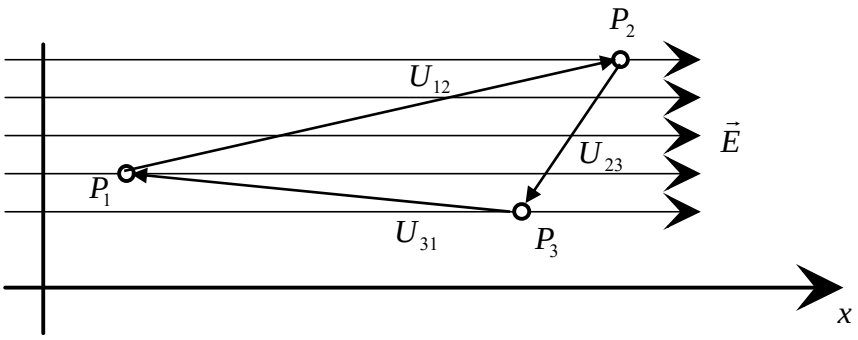


4.2.2 Generalized Kirchhoff's voltage law

We know: In a mesh the sum of all voltages is 0:

$$U_{12} + U_{23} + U_{31} = 0$$

If the voltages are replaced by the integrals over the electric field strength, the result is:

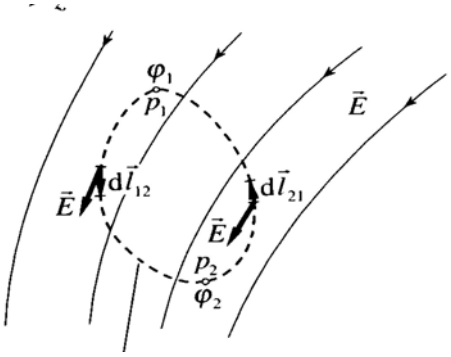


$$\int_{P_1}^{P_2} \vec{E} \cdot d\vec{l} + \int_{P_2}^{P_3} \vec{E} \cdot d\vec{l} + \int_{P_3}^{P_1} \vec{E} \cdot d\vec{l} = \oint \vec{E} \cdot d\vec{l} = 0$$

Generalization:

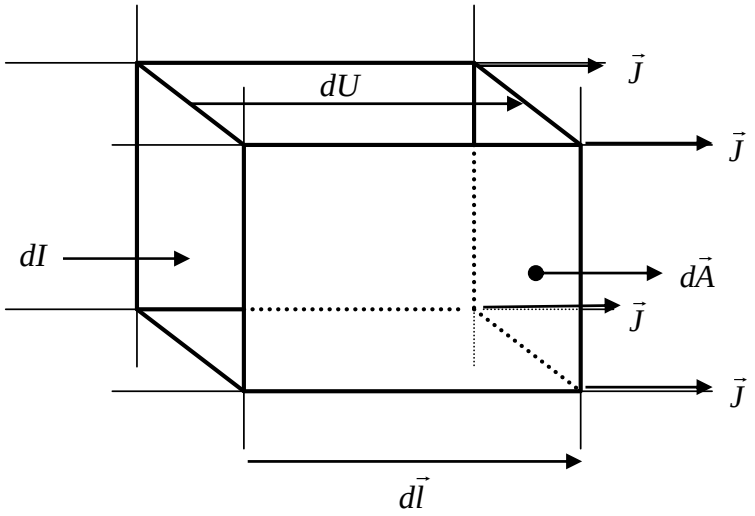
In a stationary (time independent) electric field, the total voltage along a closed curve is 0.

$$\boxed{\oint \vec{E} \cdot d\vec{l} = 0}$$



The electric current density follows the potential slope. Therefore, the electric field strength $\vec{E}$ and the current density $\vec{J}$ have the same direction [1].

4.2.3 Ohm's Law



Ohm's law will now be applied to a square created by:

- 2 equipotential surfaces
- 4 current lines

In the inhomogeneous field, such a body turns into a three-dimensional rectangular prism when its dimensions become infinite small.

The following equations apply:

$$dU = R \cdot dI$$

$$dU = \vec{E} \cdot d\vec{l} = E \cdot dl$$

$$dI = \vec{J} \cdot d\vec{A} = J \cdot dA$$

$\Leftarrow$

One can consider:

$d\vec{l}$ points in the direction of $\vec{J}$

$\vec{J}$ and $\vec{E}$ point the same direction

$d\vec{A}$ is perpendicular to the equipotential surface

This means: $\vec{J}$, $\vec{E}$, $d\vec{l}$, $d\vec{A}$ all point in the same direction

We can therefore write:

$$R = \frac{\rho \cdot dl}{dA} = \frac{dl}{\kappa \cdot dA}$$

$$\Rightarrow \quad E \cdot dl = \frac{\rho \cdot dl}{dA} \cdot J \cdot dA$$

$$\Rightarrow \quad E = \rho \cdot J$$

ρ: resistivity (not to be confused with resistance)
κ: conductivity

In general, the relationships can be formulated thus:

$$\boxed{\vec{E} = \rho \cdot \vec{J}} \quad \text{or} \quad \boxed{\vec{J} = \kappa \cdot \vec{E}}$$

This is a generalized, differential representation of Ohm's law.

4.3 Technical current field, Resistance calculations

In the following, the resistance of different, technically interesting configurations (current fields) is calculated. If a conductance is formulated as the reciprocal value of the resistance, the duality to the electrostatic field (chapter 3.6) becomes obvious.

4.3.1 Methods for resistance calculations

There are two principle methods to calculate the resistance:

- impressed current *I*
- impressed voltage *U*.

Method *I-J-E-U-R*

With this method, a test current *I* is impressed and the resulting decrease in voltage *U* is determined. This method is useful when the electric current density has the same magnitude everywhere on an equipotential surface and points in one direction of the chosen coordinate system.

I → specify test current

J → symmetry → $\boxed{\vec{J} = \frac{I}{\vec{A}}}$ ($\left|\vec{J}\right|$ constant on equipotential surface *A*)

E → $\vec{E} = \frac{\vec{J}}{\kappa}$

U → $U = \int_s \vec{E} \cdot d\vec{s}$ integration along field line *s*

R → $R = \frac{U}{I}$

Example: Homogenous field, Cylindrical field, Spherical field (4.3.2 – 4.3.4), Series connection of resistors (4.3.5)

Method *U-E-J-I-R*

With this method, a test voltage *U* is impressed on the resistor and the resulting current *I* is determined. This method is useful when the electric field strength has the same magnitude everywhere along a field line and points in one direction of the chosen coordinate system.

U → specify test voltage

E → symmetry → $\boxed{\vec{E} = -\text{grad}\varphi}$ ($|\vec{E}|$ constant along the field line *s*)

J → $\vec{J} = \kappa \cdot \vec{E}$

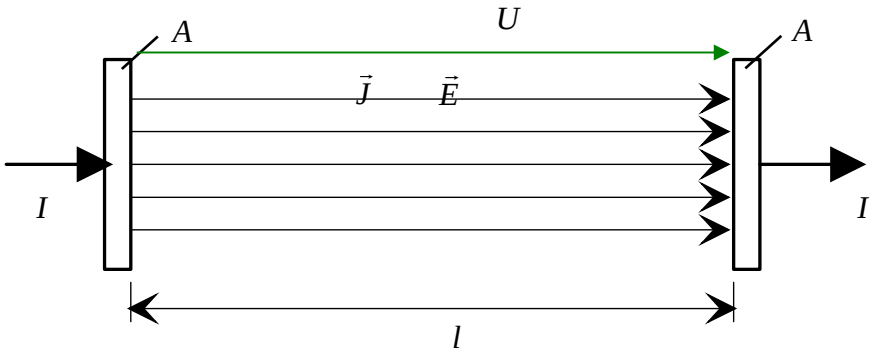
I → $I = \int_A \vec{J} \cdot d\vec{A}$ Integration through the equipotential surface *A*

R → $R = \frac{U}{I}$

Example: Homogenous field (4.3.2), Parallel connection of resistors (4.3.6)

4.3.2 Homogeneous Field

In the homogenous field, neither the current density $\vec{J}$ nor the electric field strength $\vec{E}$ are location dependent. Therefore, both methods of calculating resistance can be applied.



two possible calculations

- Starting from the current *I*
(Method *I-J-E-U-R*)

$I = \int \vec{J} \cdot d\vec{A} = J \cdot A$ $\quad J = \frac{I}{A}$

$\vec{E} = \rho \cdot \vec{J}$ $\quad E = \rho \cdot \frac{I}{A}$

$U = \int \vec{E} \cdot d\vec{l} = E \cdot l$ $\quad U = \frac{\rho \cdot l}{A} \cdot I$

$R = \frac{U}{I}$ $\quad R = \frac{\rho \cdot l}{A}$

- starting from voltage *U*
(Method *U-E-J-I-R*)

$U = \int \vec{E} \cdot d\vec{l} = E \cdot l$ $\quad E = \frac{U}{l}$

$\vec{J} = \kappa \cdot \vec{E}$ $\quad J = \frac{\kappa \cdot U}{l}$

$I = \int \vec{J} \cdot d\vec{A} = J \cdot A$ $\quad I = \frac{\kappa \cdot A}{l} \cdot U$

$R = \frac{U}{I}$ $\quad R = \frac{l}{\kappa \cdot A}$

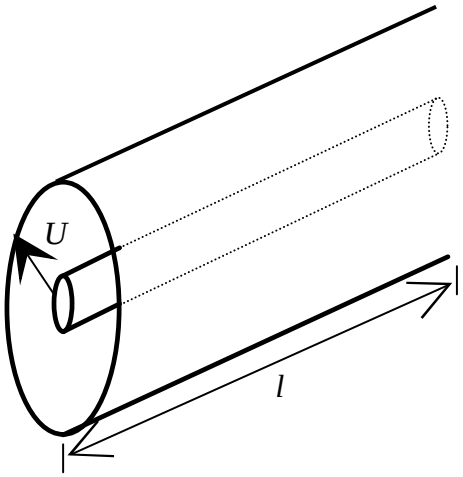
$E = \rho \cdot \frac{I}{A}$, $I = \frac{U}{R}$, $R = \frac{\rho \cdot l}{A}$

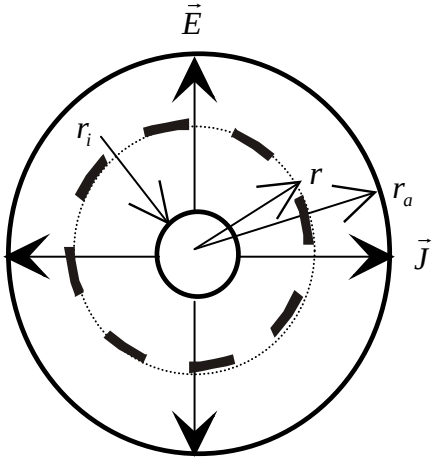
$E(U) = \frac{\rho \cdot U \cdot A}{A \cdot \rho \cdot l} \quad \Rightarrow \quad \underline{\underline{E(U) = \frac{U}{l}}}$

$J = \frac{\kappa \cdot U}{l}$, $U = R \cdot I$, $R = \frac{l}{\kappa \cdot A}$

$J(I) = \frac{\kappa \cdot l \cdot I}{l \cdot \kappa \cdot A} \quad \Rightarrow \quad \underline{\underline{J(I) = \frac{I}{A}}}$

4.3.3 Cylindrical field





Constant current density on the equipotential surfaces

→ calculation: Method *I-J-E-U-R*

$$I = \int \vec{J} \cdot d\vec{A} = J \cdot 2\pi \cdot r \cdot l \qquad \rightarrow \qquad J = \frac{I}{2\pi \cdot l} \cdot \frac{1}{r}$$

$$\vec{E} = \frac{1}{\kappa} \cdot \vec{J} \qquad \rightarrow \qquad E = \frac{I}{2\pi \cdot \kappa \cdot l} \cdot \frac{1}{r}$$

$$U = \int \vec{E} \cdot d\vec{s} = \int_{r_i}^{r_a} E \cdot dr = \frac{I}{2\pi \cdot \kappa \cdot l} \cdot \int_{r_i}^{r_a} \frac{1}{r} \cdot dr = \frac{I}{2\pi \cdot \kappa \cdot l} \cdot \ln r \Big|_{r_i}^{r_a} \qquad \rightarrow \qquad U = \frac{I}{2\pi \cdot \kappa \cdot l} \cdot \ln\left(\frac{r_a}{r_i}\right)$$

$$R = \frac{U}{I} \qquad \rightarrow \qquad R = \frac{1}{2\pi \cdot \kappa \cdot l} \cdot \ln\left(\frac{r_a}{r_i}\right)$$

Insert: $E = \frac{I}{2\pi \cdot \kappa \cdot l} \cdot \frac{1}{r}$ $\qquad I = \frac{U}{R}$ $\qquad R = \frac{1}{2\pi \cdot \kappa \cdot l} \cdot \ln\left(\frac{r_a}{r_i}\right)$

$$E(U) = \frac{U}{2\pi \cdot \kappa \cdot l \cdot \frac{1}{2\pi \cdot \kappa \cdot l} \cdot \ln\left(\frac{r_a}{r_i}\right)} \cdot \frac{1}{r} \qquad \Rightarrow \qquad \underline{\underline{E(U) = \frac{U}{\ln\left(\frac{r_a}{r_i}\right)} \cdot \frac{1}{r}}}$$

The maximum value of E occurs for $r = r_i$ → $E_{\max} = \frac{U}{\ln\left(\frac{r_a}{r_i}\right)} \cdot \frac{1}{r_i}$

Let us formulate an additional question: How large must be r_i, if the external radius r_a is predefined and the voltage U is constant, so E_{max} will adopt the smallest possible value? This is called the optimal relation of radii.

Procedure: We create the derivative of E_{max} to r_i. The derivation must be 0 for an extremum.

$$\frac{dE_{max}}{dr_i} = U \cdot \frac{d}{dr_i}\left(\frac{1/r_i}{\ln\left(r_a/r_i\right)}\right)$$

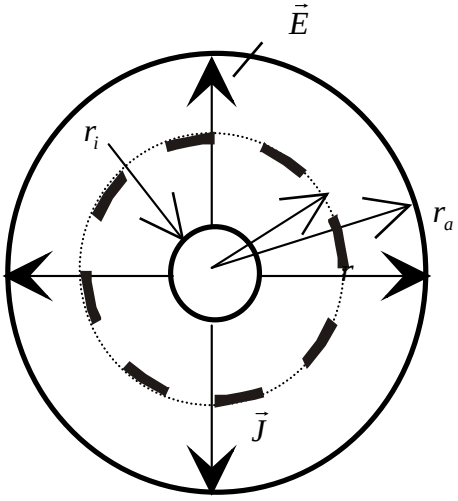
Quotient-equation for derivations: $\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$

$$\frac{dE_{max}}{dr_i} = U \cdot \frac{-\frac{1}{r_i^2}\cdot\ln\left(\frac{r_a}{r_i}\right) - \frac{1}{r_i}\cdot\frac{1}{r_a/r_i}\cdot\left(-\frac{r_a}{r_i^2}\right)}{\left[\ln\left(\frac{r_a}{r_i}\right)\right]^2} = \frac{U}{r_i^2\cdot\left[\ln\left(\frac{r_a}{r_i}\right)\right]^2}\cdot\left[1-\ln\left(\frac{r_a}{r_i}\right)\right]$$

$$\frac{dE_{max}}{dr_i} = 0 \quad \rightarrow \quad 1-\ln\left(\frac{r_a}{r_i}\right) = 0 \quad \rightarrow \quad \ln\left(\frac{r_a}{r_i}\right) = 1 \quad \rightarrow$$

In order to get the best radius ratio, the following equation must be applicable: $\frac{r_a}{r_i} = e = 2{,}718$

4.3.4 Spherical field



Constant electric-density on the equipotencial surface

→ Calculation: Method *I-J-E-U-R*

$$I = \int \vec{J}\cdot d\vec{A} = J\cdot 4\pi\cdot r^2 \quad \rightarrow \quad J = \frac{I}{4\pi}\cdot\frac{1}{r^2}$$

$$\vec{E} = \frac{1}{\kappa}\cdot\vec{J} \quad \rightarrow \quad E = \frac{I}{4\pi\cdot\kappa}\cdot\frac{1}{r^2}$$

$$U = \int \vec{E}\cdot d\vec{s} = \int_{r_i}^{r_a} E\cdot dr = \frac{I}{4\pi\cdot\kappa}\cdot\int_{r_i}^{r_a}\frac{1}{r^2}\cdot dr = \frac{I}{4\pi\cdot\kappa}\cdot\left(-\frac{1}{r}\right)\Big|_{r_i}^{r_a} = \frac{I}{4\pi\cdot\kappa}\cdot\left(-\frac{1}{r_a}+\frac{1}{r_i}\right)$$

$$\rightarrow \quad U = \frac{I}{4\pi\cdot\kappa}\cdot\frac{r_a - r_i}{r_a\cdot r_i}$$

$$R = \frac{U}{I} \quad \rightarrow \quad R = \frac{1}{4\pi\cdot\kappa}\cdot\frac{r_a - r_i}{r_a\cdot r_i}$$

Insert: $E = \frac{I}{4\pi \cdot \kappa} \cdot \frac{1}{r^2}$ $I = \frac{U}{R}$ $R = \frac{1}{4\pi \cdot \kappa} \cdot \frac{r_a - r_i}{r_a \cdot r_i}$

$$E(U) = \frac{U}{4\pi \cdot \kappa \cdot \frac{1}{4\pi \cdot \kappa} \cdot \frac{r_a - r_i}{r_a \cdot r_i}} \cdot \frac{1}{r^2} \quad \Rightarrow \quad \underline{\underline{E(U) = \frac{r_a \cdot r_i}{r_a - r_i} \cdot \frac{U}{r^2}}}$$

One important application of the spherical field is the application for hemispheres. A hemispherical ground electrode will be used as an example.

This involves half of a spherical field with $r_a \to \infty$

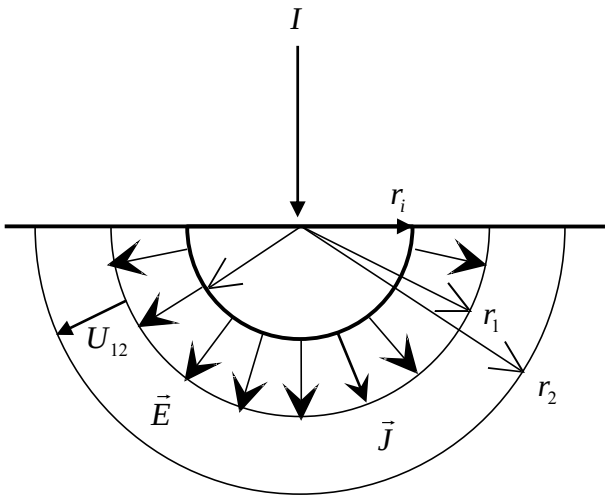
$$R = 2 \cdot \lim_{r_a \to \infty} \frac{1}{4\pi \cdot \kappa} \cdot \frac{r_a - r_i}{r_a \cdot r_i}$$

$$R = \frac{1}{2\pi \cdot \kappa} \cdot \lim_{r_a \to \infty} \cdot \frac{r_a}{r_a \cdot r_i}$$

It is now possible to determine the transition resistance to the 'far-away earth' ($r_a \to \infty$):

$$R = \frac{1}{2\pi \cdot \kappa \cdot r_i} = \frac{\rho}{2\pi \cdot r_i}$$

Because only half of the cross sectional area is available for the power feed, $\vec{J}$ and $\vec{E}$ are twice as large for the same current *I* as in a (full)-spherical field.



$$\underline{\underline{J = \frac{I}{2\pi} \cdot \frac{1}{r^2}}} \qquad \underline{\underline{E = \frac{\rho \cdot I}{2\pi} \cdot \frac{1}{r^2}}}$$

The voltage between two points, with the distances r_1 and r_2 from the centre of the (hemi-) sphere can be calculated as follows:

$$U_{12} = \int_{r_1}^{r_2} E \cdot dr = \frac{\rho \cdot I}{2\pi} \cdot \int_{r_1}^{r_2} \frac{1}{r^2} \cdot dr = \frac{\rho \cdot I}{2\pi} \cdot \left(-\frac{1}{r} \right)_{r_1}^{r_2} = \frac{\rho \cdot I}{2\pi} \cdot \frac{r_2 - r_1}{r_2 \cdot r_1}$$

For $\Delta r = r_2 - r_1 << r_1, r_2$: $\quad U_{12} \approx \frac{\rho \cdot I}{2\pi} \cdot \frac{\Delta r}{r^2}$

In the personal safety industry, the step potential of Δr = 1 m is often used (typical leg spacing of a person).

Numerical example: Soil: $\rho \approx 100\ \Omega$m, I = 100 kA, r = 10 m, Δr = 1 m

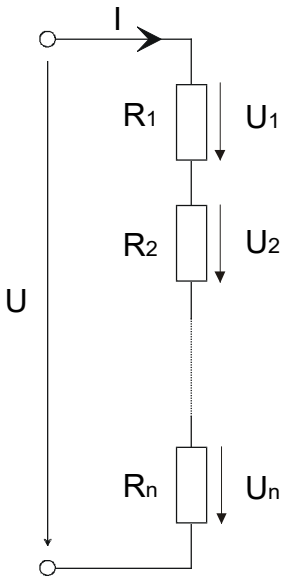
$$U_{12} \approx \frac{100\Omega m \cdot 100 \cdot 10^3\, A}{2\pi} \cdot \frac{1m}{100 m^2} = 15{,}9kV$$

4.3.5 Series connection of resistors

Method *I-J-E-U-R*

$\oint_{A_\nu} \vec{J} \cdot d\vec{A} = 0$ The same current flows through every cross-sectional area.

Equivalent circuit



$$U = U_1 + U_2 + \ldots + U_n$$
$$U = R_1 \cdot I + R_2 \cdot I + \ldots + R_n \cdot I$$
$$R = \frac{U}{I} = R_1 + R_2 + \ldots + R_n$$

$$\boxed{R = \sum_{\nu=1}^{n} R_\nu}$$

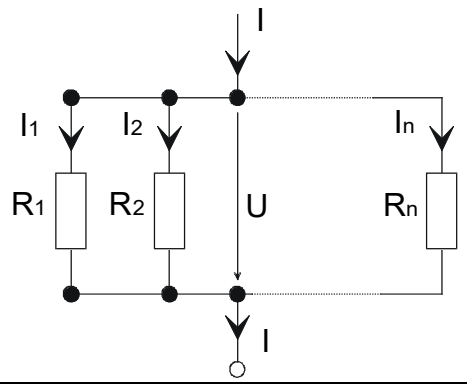
Resistors are added in a series connection

4.3.6 Parallel connection of resistors

Method *U-E-J-I-R*

$\oint_{s_\nu} \vec{E} \cdot d\vec{s} = 0$ The potential difference between the two nodes is the same.

Equivalent circuit



Qualitative:
The electric charges can use more paths parallel to the smallest of the *n* resistors. The total resistance is smaller than the smallest individual resistance.

$$I = I_1 + I_2 + ... + I_n$$

$$I = \frac{U}{R_1} + \frac{U}{R_2} + ... + \frac{U}{R_n}$$

$$G = \frac{1}{R} = \frac{I}{U} = \frac{1}{R_1} + \frac{1}{R_2} + ... + \frac{1}{R_n}$$

$$\boxed{\frac{1}{R} = \sum_{\nu=1}^{n} \frac{1}{R_\nu}} \qquad \boxed{G = \sum_{\nu=1}^{n} G_\nu}$$

The conductances are added in the parallel series.

4.3.7 Summary

The following solution strategy is useful to calculate resistances:

1. Choose a proper coordinate system

2. Heed the marginal conditions
   - ideal conducting electrodes ($\vec{E}$ and $\vec{J}$ are placed vertically on the metal)
   - different materials (E_t constant and J_n constant)

3. Draw field line diagram with equipotential surfaces

4. Consideration of symmetry

   a) If the electric current density $\vec{J}$ on an equipotential surface has to be equal everywhere

   ⇨ Method ***I-J-E-U-R***, this corresponds to a **series connection** of infinitesimal resistors

   $$R = \int dR = \frac{1}{I} \int dU = \int \frac{dl}{\kappa \cdot A}$$

   b) If the electric field strength $\vec{E}$ must be equal everywhere along a field line

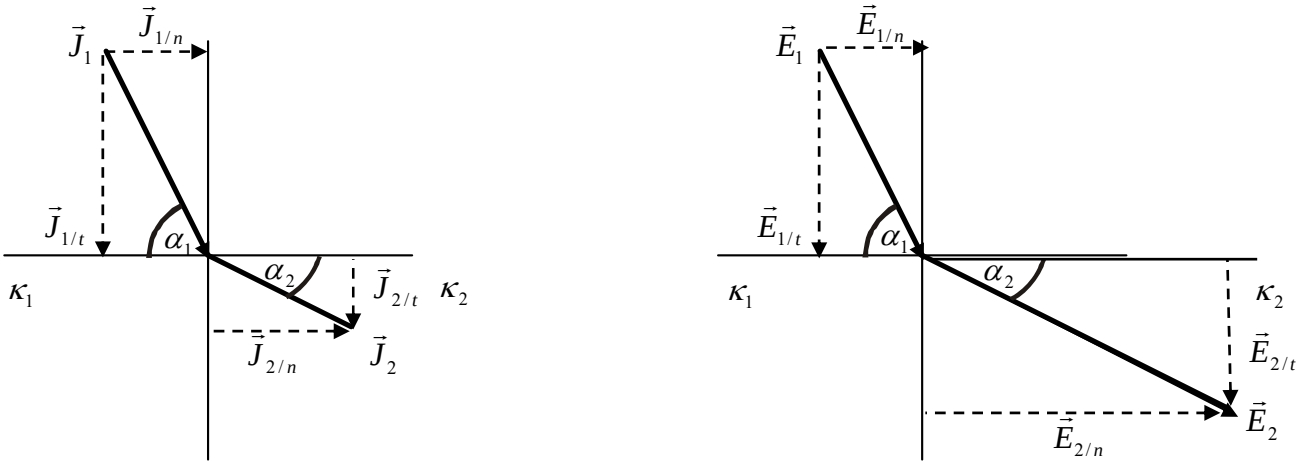
   ⇨ Method ***U-E-J-I-R***, this corresponds to the **parallel connection** of infinitesimal resistors

   $$\frac{1}{R} = \int d\frac{1}{R} = \frac{1}{U} \int dI = \int \frac{\kappa \cdot dA}{l}$$

4.4 Boundary conditions

Here, as in the electrostatic field (3.8), similar conditions as in optics apply. In the following, we will examine the boundary between two materials, which have different conductivities. Furthermore, we assume that the field lines, as shown in the diagram, do not intersect the boundary vertically.

Because of the different conductivities, the amount and direction of the field strength in both materials will not be the same. However, the conditions of the irrotationality of the electrical field (generalized Kirchhoff's second law) and the generalized Kirchhoff's first law also have to be fulfilled at the boundary.



$\kappa_1 \neq \kappa_2 \qquad here: \kappa_1 > \kappa_2$

Again we will examine an element with infinite small volume (a disc) directly on the borderline between both materials (see also 3.8):

Material equation:

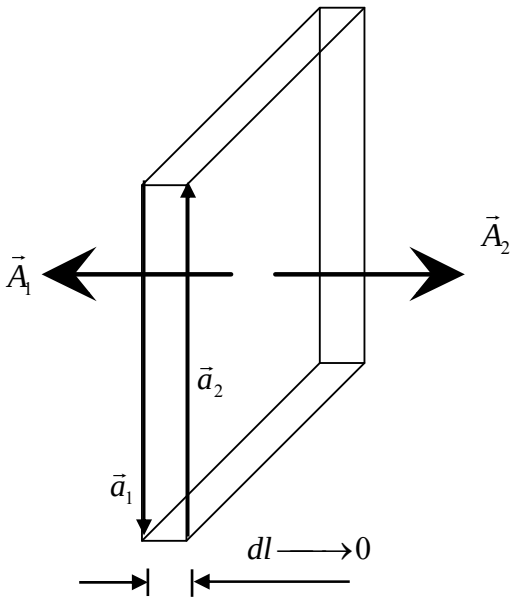
$$\vec{J} = \kappa \cdot \vec{E}$$

Irrotationality:

$$\oint_s \vec{E} \cdot d\vec{s} = 0$$

Node equation:

$$\oint_A \vec{J} \cdot d\vec{A} = 0$$



$$\left|\vec{a}_1\right| = \left|\vec{a}_2\right| = a$$

$$\left|\vec{A}_1\right| = \left|\vec{A}_2\right| = A$$

A, a: both infinite small

$$\oint_s \vec{E} \cdot d\vec{s} = \vec{E}_1 \cdot \vec{a}_1 + \vec{E}_2 \cdot \vec{a}_2 = E_{1/t} \cdot a - E_{2/t} \cdot a = 0$$

$$\boxed{E_{1/t} = E_{2/t}}$$

The tangential component of the electric field strength is constant

$$\oint_A \vec{J} \cdot d\vec{A} = \vec{J}_1 \cdot \vec{A}_1 + \vec{J}_2 \cdot \vec{A}_2 = -J_{1/n} \cdot A + J_{2/n} \cdot A = 0$$

$$\boxed{J_{1/n} = J_{2/n}}$$

The normal component of the current density is constant

From the combination of both relationships ensues:

$$J_{1/n} = J_{2/n} \quad \rightarrow \quad \kappa_1 \cdot E_{1/n} = \kappa_2 \cdot E_{2/n}$$

$$E_{1/t} = E_{2/t} \quad \Rightarrow \quad \frac{1}{\kappa_1} \cdot \frac{E_{1/t}}{E_{1/n}} = \frac{1}{\kappa_2} \cdot \frac{E_{2/t}}{E_{2/n}}$$

$$\frac{1}{\kappa_1} \cdot \tan\alpha_1 = \frac{1}{\kappa_2} \cdot \tan\alpha_2$$

$$\boxed{\frac{\tan\alpha_1}{\tan\alpha_2} = \frac{\kappa_1}{\kappa_2}}$$

Let us discuss another special case here: How will the electrical field lines and the current lines look like, when exiting an ideal conductor with $\kappa_1 \rightarrow \infty$? Here, the above shown relationship results in:

$$\frac{\tan\alpha_1}{\tan\alpha_2} = \frac{\kappa_1}{\kappa_2} \rightarrow \infty$$

For this there are now two possibilities:

$\tan\alpha_1 \rightarrow \infty \quad \alpha_1 = 90°$

The current lines run parallel to the boundaries. They do not enter into the surrounding, less conducting material with κ_2.

$\tan\alpha_2 = 0 \quad \alpha_2 = 0°$

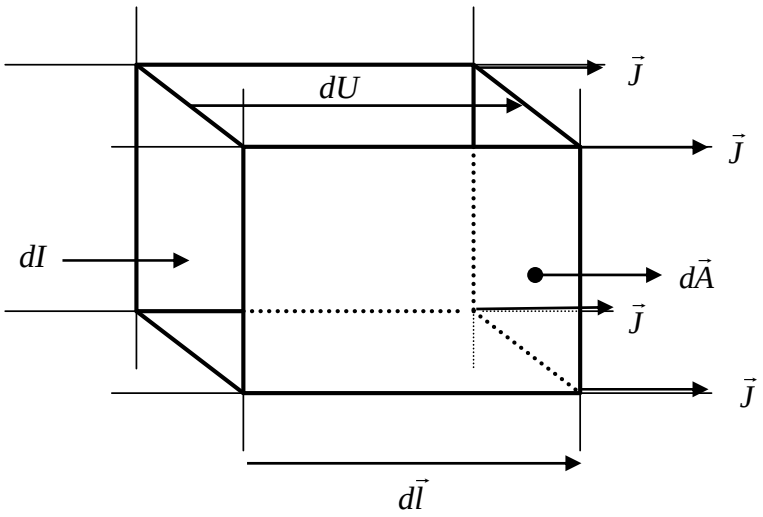
The current lines leave the conductor at a right angle to the boundary. This happens at the end of the conductor if the second material with the smaller conductivity κ_2 is attached there. The same then also applies to the electric field lines, which also leave the conductor there at a right angle to the boundary.

A further result from $J_{1t} = \kappa_1 \cdot E_{1t}$ and for finite current density is, that the tangential field strength on the inside of the ideal conductor has to be 0. This means that the potential is the same everywhere within an ideal conductor.

4.5 Power density and Joule's power

The power density indicates the energy per volume and time, which is released through current transfer (moved charge carrier) to the crystal lattice. This is the kinetic energy of the charge carrier transformed into heat by the electric field. Since this energy is taken from the electric field, it is known as Joule's power loss.

Power Density



This is the power which is transformed into an infinite small volume. For this, we reexamine the rectangular prism from 4.2.3.

$$dP = dU \cdot dI = E \cdot dl \cdot J \cdot dA = E \cdot J \cdot dl \cdot dA = E \cdot J \cdot dV$$

$$\frac{dP}{dV} = E \cdot J = \rho \cdot J^2 = \kappa \cdot E^2$$ Power density = Power per volume element

Unit: $\left[\frac{dP}{dV}\right] = \frac{W}{m^3}$

Cylindrical field (4.3.3)

The power density in a cylindrical field amounts to:

$$\frac{dP}{dV} = J \cdot E = \kappa \cdot E^2 \qquad E = \frac{U}{\ln\left(\frac{r_a}{r_i}\right)} \cdot \frac{1}{r} \qquad \rightarrow \qquad \underline{\underline{\frac{dP}{dV} = \frac{\kappa \cdot U^2}{\left[\ln\left(\frac{r_a}{r_i}\right)\right]^2} \cdot \frac{1}{r^2}}}$$

Joule's power loss *P* in a cylindrical field can be calculated through integration of the field volume:

$$P = \int \frac{dP}{dV} \cdot dV = \int_{r_i}^{r_a} \frac{dP}{dV} \cdot 2\pi \cdot l \cdot r \cdot dr = 2\pi \cdot l \cdot \frac{\kappa \cdot U^2}{\left[\ln\left(\frac{r_a}{r_i}\right)\right]^2} \cdot \int_{r_i}^{r_a} \frac{r}{r^2} \cdot dr = 2\pi \cdot l \cdot \frac{\kappa \cdot U^2}{\left[\ln\left(\frac{r_a}{r_i}\right)\right]}$$

Spherical field (4.3.4)

The power density in the spherical field amounts to:

$$\frac{dP}{dV} = \kappa \cdot E^2 \qquad E = \frac{r_a \cdot r_i}{r_a - r_i} \cdot \frac{U}{r^2} \qquad \rightarrow \qquad \underline{\underline{\frac{dP}{dV} = \left(\frac{r_a \cdot r_i}{r_a - r_i} \right)^2 \cdot \frac{\kappa \cdot U^2}{r^4}}}$$

The power density decreases by $1/r^4$.

Joule's power loss P in a spherical field can be calculated through integration of the field volume:

$$P = \int \frac{dP}{dV} \cdot dV = \int_{r_i}^{r_a} \frac{dP}{dV} \cdot 4\pi \cdot r^2 \cdot dr = 4\pi \cdot \left(\frac{r_a \cdot r_i}{r_a - r_i} \right)^2 \cdot \kappa \cdot U^2 \cdot \int_{r_i}^{r_a} \frac{r^2}{r^4} \cdot dr =$$

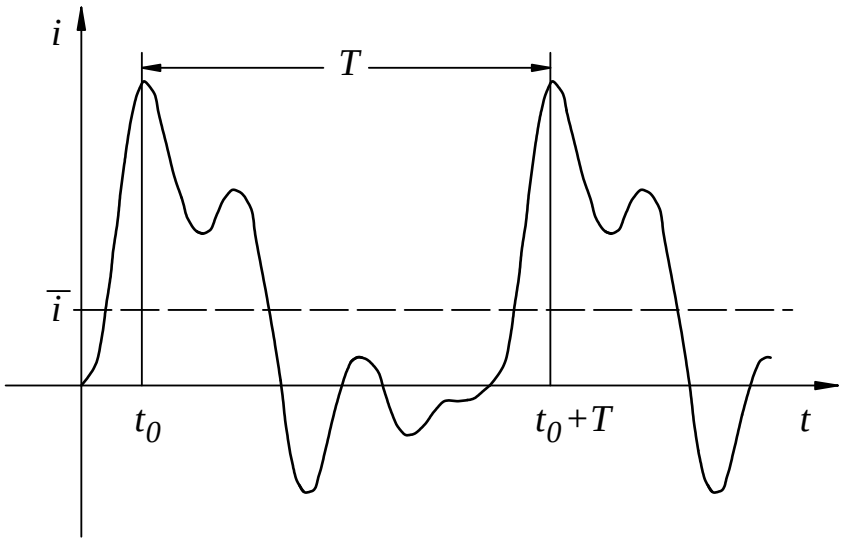
$$= 4\pi \cdot \left(\frac{r_a \cdot r_i}{r_a - r_i} \right)^2 \cdot \kappa \cdot U^2 \cdot \left[\frac{-1}{r} \right]_{r_i}^{r_a} = 4\pi \cdot \left(\frac{r_a \cdot r_i}{r_a - r_i} \right)^2 \cdot \kappa \cdot U^2 \cdot \left[\frac{1}{r_i} - \frac{1}{r_a} \right] = 4\pi \cdot \left(\frac{r_a \cdot r_i}{r_a - r_i} \right) \cdot \kappa \cdot U^2$$

5. Alternating Current Circuits

5.1 Periodic quantities

5.1.1 Fundamentals

A quantity is called periodic, if it‘s variation shown on a graph to a base of time repeats periodically at equal intervals. The next figure is an example of a periodic quantity [1].



Each of these repetitions is called a **cycle**. The periodicity of a current or a voltage can be described mathematically by:

$$i(t) = i(t + nT)$$
$$u(t) = u(t + nT)$$

with: *n* any integer number
T **period** of on cycle.

The number of cycles per unit time is called the **frequency** of the periodic quantity:

$$f = \frac{1}{T}$$

The variation of the periodic quantity with time is called it‘s **waveform**.

5.1.2 Definition of an alternating quantity

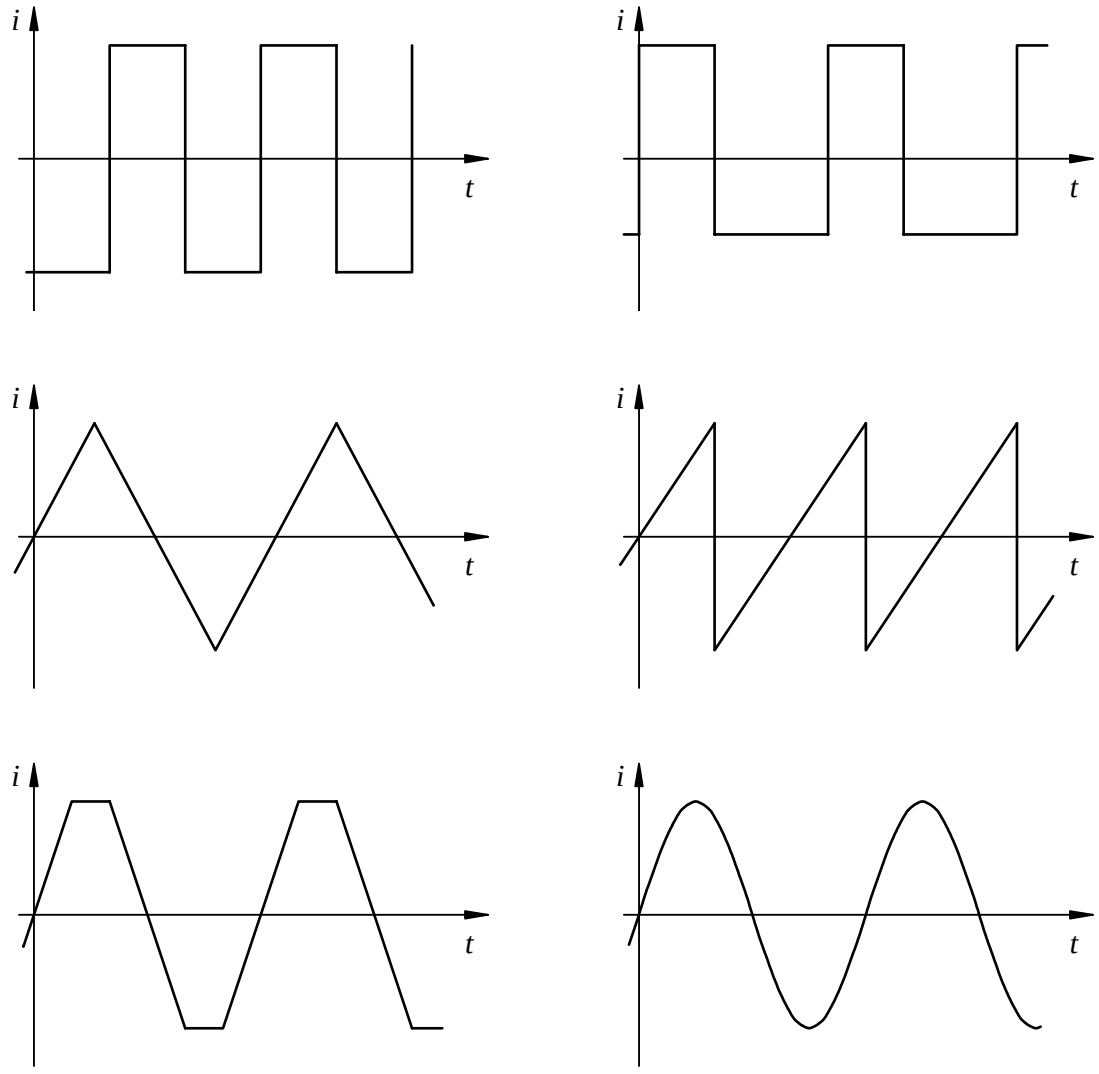
As defined before, an alternating quantity is a periodic quantity of arbitrary waveform:

$$i(t) = i(t + nT)$$
$$u(t) = u(t + nT)$$

However, there is a second condition for an alternating quantity: the arithmetic mean value must be zero:

$$\bar{i} = \frac{1}{T} \cdot \int_{t_0}^{t_0+T} i(t)dt = 0$$
$$\bar{u} = \frac{1}{T} \cdot \int_{t_0}^{t_0+T} u(t)dt = 0$$

The following figures show some examples for alternating currents. The waveforms are usually (one exception) symmetric to the horizontal *t*-axis.



5.1.3 Rectified mean value of an alternating quantity

Whereas per definition the mean value of an alternating quantity is zero, another mean value is used for a characterization of an alternating quantity. This is the rectified mean value, i.e. first the signal is rectified, then the mean value is taken.

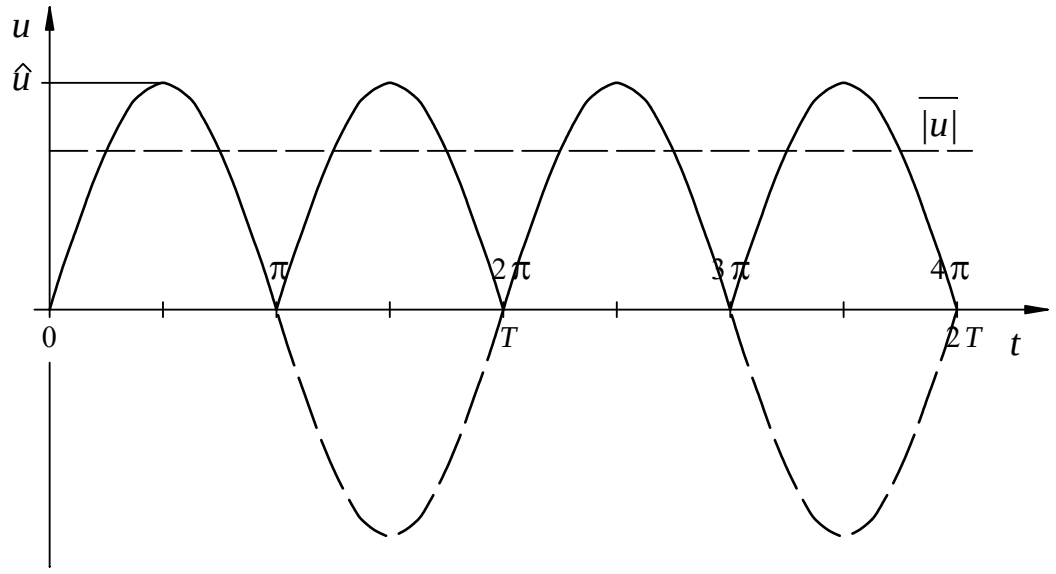
The response of moving coil instruments is proportional to the arithmetic mean of a current or voltage. For alternating currents and voltages their reading is zero. AC voltages and currents therefore have to be rectified before being applied to the moving coil.

Remark: AC or ac is the abbreviation for alternating current. However, it indicates also the alternating nature of other electrical quantities, e.g. an alternating voltage. So all electrical quantities being alternating are marked with AC or ac (AC current, AC voltage, AC circuit, AC power, AC impedance, etc.).

For a full-wave rectification (see next figure) the instrument‘s reading will then be proportional to the rectified mean value (here for example a rectified sinusoidal voltage):

$$\overline{|u|} = \frac{1}{T} \cdot \int_{t_0}^{t_0+T} |u(t)| dt$$

For a sinusoid of frequency *f* the time axis can also be scaled in terms of the rotation angle with a value of 2π for one cycle (one full rotation).



So a value of 2π corresponds to the period *T*. If we introduce the angular frequency

$$\omega = 2\pi \cdot f$$

we can denote a sinusoidal voltage as a function of time:

$$u(t) = \hat{u} \cdot \sin(\omega t)$$

The rectified mean value can then be calculated:

$$\overline{|u|} = \frac{1}{T} \int_{t_0}^{t_0+T} |\hat{u} \cdot \sin(\omega t)| dt$$

$$\overline{|u|} = \frac{1}{T} \left\{ \int_{0}^{\frac{1}{2}T} \hat{u} \cdot \sin(\omega t) dt + \int_{\frac{1}{2}T}^{T} \hat{u} \cdot [-\sin(\omega t)] dt \right\}$$

$$\overline{|u|} = \frac{\hat{u}}{T} \left\{ -\frac{1}{\omega} \cdot \cos(\omega t) \Big|_{t=0}^{T/2} + \frac{1}{\omega} \cdot \cos(\omega t) \Big|_{t=T/2}^{T} \right\}$$

$$\overline{|u|} = \frac{\hat{u}}{\omega T} \left\{ -\cos\left(\frac{\omega T}{2}\right) - (-1) + \cos(\omega T) - \cos\left(\frac{\omega T}{2}\right) \right\}$$

$$\overline{|u|} = \frac{\hat{u}}{\frac{2\pi}{T} T} \{ -\cos(\pi) + 1 + \cos(2\pi) - \cos(\pi) \}$$

$$\overline{|u|} = \frac{\hat{u}}{2\pi} \{1 + 1 + 1 + 1\}$$

$$\overline{|u|} = \frac{2}{\pi} \cdot \hat{u} = 0{,}637 \cdot \hat{u}$$

Strictly speaking the rectified sinusoid is periodic with a period of *T*/2. Moreover each of these cycles is symmetrical to its peak value. So the integration period can be reduced to *T*/4:

$$\overline{|u|} = \frac{4}{T} \int_{0}^{T/4} \hat{u} \cdot \sin(\omega t) dt$$

$$\overline{|u|} = \frac{4 \cdot \hat{u}}{T} \left\{ -\frac{1}{\omega} \cdot \cos(\omega t) \Big|_{t=0}^{T/4} \right\}$$

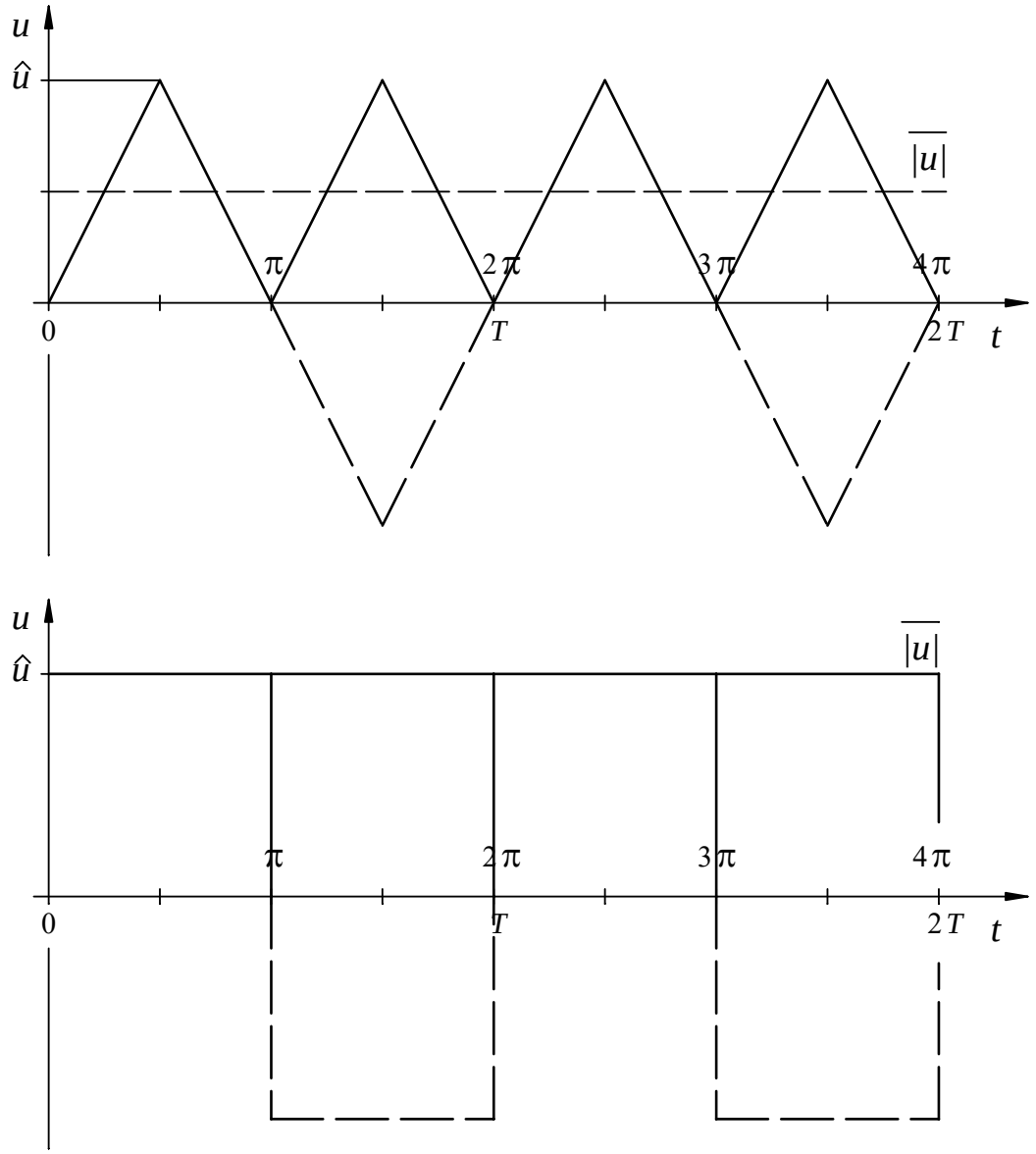
$$\overline{|u|} = \frac{4 \cdot \hat{u}}{\omega T} \left\{ -\cos\left(\frac{\omega T}{4}\right) - (-1) \right\}$$

$$\overline{|u|} = \frac{4 \cdot \hat{u}}{2\pi} \{0 + 1\} = \frac{2}{\pi} \cdot \hat{u}$$

The following two figures show two more examples for a rectified mean value:

- Triangular waveform: $\overline{|u|} = \frac{1}{2} \cdot \hat{u}$

- Square wave: $\overline{|u|} = \hat{u}$



5.1.4 Root mean square (RMS) value of an alternating quantity

The root mean square value (RMS-value) of an alternating voltage or current is defined in a way that it is proportional to the power dissipated, if it is applied to a resistor. The background is that an AC voltage or current with a given RMS-value should be exactly as efficient from a physical point-of-view like a DC current or voltage with the same (constant) value.

RMS-value of a current (similar definition is valid for a voltage):

$$I = \sqrt{\frac{1}{T}\int_{t_0}^{t_0+T} i^2(t)\,dt}$$

An alternating current (or voltage) with an RMS-value of *I* leads to the same (physical) power dissipation in an ohmic resistor as a direct current (or voltage) of the same value *I*

$$\overline{p} = P = \frac{1}{T}\int_{t_0}^{t_0+T} p(t)dt = \frac{1}{T}\int_{t_0}^{t_0+T} R \cdot i^2(t)dt = \frac{R}{T}\int_{t_0}^{t_0+T} i^2(t)dt = R \cdot I^2$$

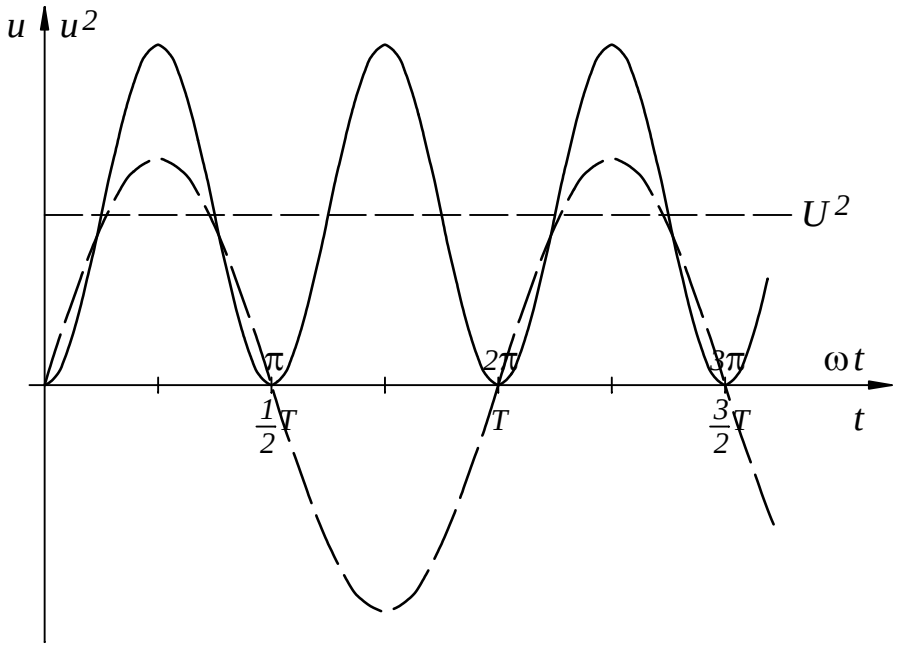
By means of the RMS-value e.g. the heating effect of an AC current can be compared to that of a DC current.

Moving iron instruments as well as electrodynamic instruments give a reading proportional to i^2. They can be calibrated to give the RMS - value directly.

Modern so-called True-RMS-Meters convert the input quantity to a digital number and calculate the integral given above numerically to determine the true RMS-value.

Sinusoidal waveform

Below we see a sinusoidal voltage (dotted line) and the squared voltage (solid line). We may assume, that the mean value U^2 of the squared voltage is half of the maximum value $\hat{u}^2$. This can be verified by calculation.



To calculate the RMS-value of a sinusoidal voltage (dotted line), we have to integrate the square of this voltage (solid line). As shown in the figure above, this square has a frequency twice that of the voltage itself. For symmetry reasons it is sufficient to integrate from $t = 0$ to $t = T/4$.

$$U^2 = \frac{1}{T/4} \int_0^{T/4} \hat{u}^2 \sin^2(\omega t) dt$$

Using the relation $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$ we get:

$$U^2 = \frac{4 \cdot \hat{u}^2}{2 \cdot T} \int_0^{T/4} [1 - \cos(2\omega t)] dt$$

$$U^2 = \frac{2 \cdot \hat{u}^2}{T} \left[t - \frac{1}{2 \cdot \omega} \cdot \sin(2\omega t) \right]_{t=0}^{T/4}$$

$$U^2 = \frac{2 \cdot \hat{u}^2}{T} \left[\frac{T}{4} - \frac{1}{2 \cdot \omega} \cdot \sin\left(\frac{\omega T}{2} \right) \right]$$

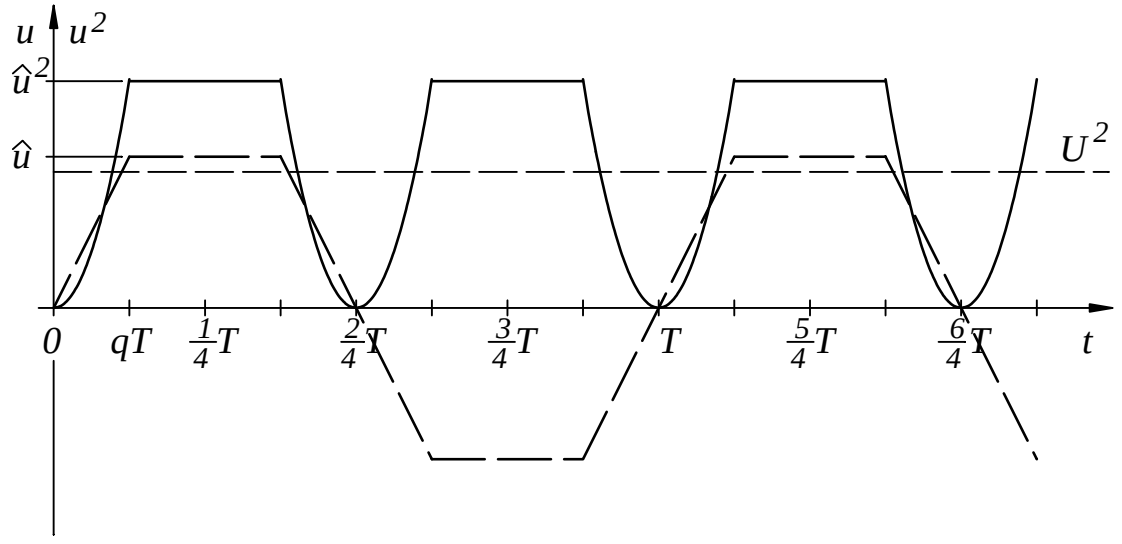
$$U^2 = \frac{2 \cdot \hat{u}^2}{T} \left[\frac{T}{4} - \frac{1}{2 \cdot \omega} \cdot \sin(\pi) \right] = \frac{\hat{u}^2}{2}$$

With that we finally get the RMS-value for a sinusoidal voltage:

$$U = \frac{\hat{u}}{\sqrt{2}} = 0{,}707 \cdot \hat{u}$$

Trapezoidal waveform

As a second example we determine a trapezoidal voltage (dotted line) and its square (solid line).



Between $t = 0$ and $t = qT$ the voltage curve is a straight line with a slope of $m = \frac{\hat{u}}{qT}$.

For symmetry reasons it is again sufficient to integrate from $t = 0$ to $t = T/4$.

$$U^2 = \frac{1}{T/4}\left\{\int_0^{qT}\left(\frac{\hat{u}}{qT}\cdot t\right)^2 dt + \int_{qT}^{T/4}\hat{u}^2 dt\right\}$$

$$U^2 = \frac{4}{T}\left\{\frac{1}{3}\cdot\frac{\hat{u}^2}{q^2\cdot T^2}\cdot t^3\Big|_{t=0}^{qT} + \hat{u}^2\cdot t\Big|_{t=qT}^{T/4}\right\}$$

$$U^2 = \frac{4\cdot\hat{u}^2}{T}\left\{\frac{1}{3}\cdot\frac{q^3\cdot T^3}{q^2\cdot T^2} + \frac{1}{4}\cdot T - q\cdot T\right\}$$

$$U^2 = 4\cdot\hat{u}^2\left\{\frac{1}{3}\cdot q + \frac{1}{4} - q\right\}$$

$$U^2 = \hat{u}^2\left(1 - \frac{8}{3}\cdot q\right)$$

And finally we get the RMS-value:

$$U = \hat{u}\cdot\sqrt{1 - \frac{8}{3}\cdot q}$$

With the trapezoidal shape we can also solve two other important cases:

- For $q = 0$ we get a square wave and the RMS-value
  $U = \hat{u}$

- For $q = 1/4$ the trapezoid becomes a triangular and we get the RMS-value
  $U = \frac{\hat{u}}{\sqrt{3}}$

5.1.5 Form factor and crest factor

A moving coil instrument as well as a simple digital multimeter switched to an AC range determines the rectified mean value of the input current or voltage. The scale is calibrated in a way, that the rectified mean value is multiplied by the so-called form factor, so that the instrument‘s reading is equivalent to the RMS-value.

The form factor *F* is defined as the ratio of the RMS-value *U* to the rectified mean value $\overline{|u|}$. It depends on the waveform of the signal, so that the instrument readings are only correct RMS-values for sinusoidal quantities.

The ratio of the largest instantaneous value of a signal $\hat{u}$ to its RMS-value U is called crest factor or peak factor. The symbol used is ξ.

waveform	form factor $F = \frac{U}{\overline{\lvert u \rvert}}$	crest factor $\xi = \frac{\hat{u}}{U}$
triangular	$F = \frac{\frac{\hat{u}}{\sqrt{3}}}{\frac{1}{2}\cdot\hat{u}} = \frac{2}{\sqrt{3}} = 1{,}155$	$\xi = \frac{\hat{u}}{\frac{\hat{u}}{\sqrt{3}}} = \sqrt{3} = 1{,}732$
sinusoidal	$F = \frac{\frac{\hat{u}}{\sqrt{2}}}{\frac{2}{\pi}\cdot\hat{u}} = \frac{\pi}{2\cdot\sqrt{2}} = 1{,}111$	$\xi = \frac{\hat{u}}{\frac{\hat{u}}{\sqrt{2}}} = \sqrt{2} = 1{,}414$
rectangular	$F = \frac{\hat{u}}{\hat{u}} = 1{,}000$	$\xi = \frac{\hat{u}}{\hat{u}} = 1{,}000$

5.1.6 Pulsating quantities

Often AC quantities are mixed with DC quantities. Such mixed quantities are called pulsating quantities:

pulsating quantity	=	AC quantity	+	DC quantity
$i(t)$	=	$i_{\sim}(t)$	+	$i_{=}(t)$

The pulsating quantity is periodic in time: $i(t) = i(t + nT)$

and its arithmetic mean value is: $i_{=} = \bar{i} = \frac{1}{T}\cdot\int_{t_0}^{t_0+T} i(t)dt \neq 0$.

The RMS-value of a pulsating current can be calculated:

$$I = \sqrt{\frac{1}{T}\int_{t_0}^{t_0+T}\left(i_{=} + i_{\sim}\right)^2 dt}$$

$$I^2 = \frac{1}{T}\int_{t_0}^{t_0+T} i_{=}^2 dt + \frac{2}{T}\int_{t_0}^{t_0+T} (i_{=} \cdot i_{\sim})dt + \frac{1}{T}\int_{t_0}^{t_0+T} i_{\sim}^2 dt$$

$$I^2 = I_{=}^2 + 0 + I_{\sim}^2$$

The RMS-value is then: $I = \sqrt{I_{=}^2 + I_{\sim}^2}$

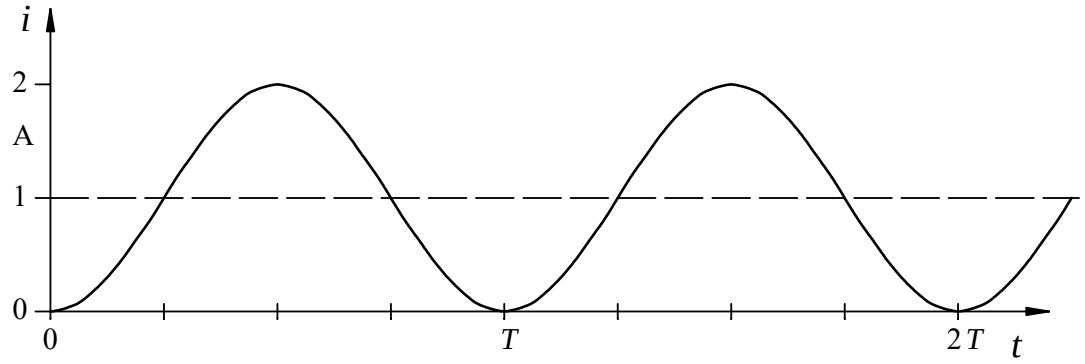
An equivalent relation can be used to add the RMS values of waves with different frequencies.

The following example demonstrates the relation. We have the following pulsating current with a general sinusoidal shape. The current can be described analytically:

$$i(t) = 1A \cdot [1 - \cos(\omega t)]$$

with: $i_{=} = 1A$ $\qquad I_{=} = 1A$

$i_{\sim} = -1A \cdot \cos(\omega t)$ $\qquad I_{\sim} = \frac{1}{\sqrt{2}} A$



The RMS-value is: $I = \sqrt{(1A)^2 + \left(\frac{1}{\sqrt{2}} A\right)^2} = \sqrt{1{,}5}A = 1{,}225\,A$

This is <u>not</u> equal to the sum of the RMS-values of the two components:

$$I \neq 1A + \frac{1}{\sqrt{2}} A = 1{,}707\,A$$

5.1.7 Fourier analysis

All periodic time functions *f(t)* can be described as a superposition of sine and cosine functions with different frequencies and different phase shift and, if necessary, a direct component. The superposition usually contains a sine wave with the basic frequency

1/*T* corresponding to the period *T* of the function f(t). This basic wave is also called first harmonic. The frequencies of the other sine waves (higher harmonics) are always integer multiples of the basic frequency. The superposition of the direct component, the basic wave and the higher harmonics is performed as a Fourier analysis.

Aim: A periodic time function *f(t)* with the angular frequency ω has to be replaced as good as possible by an approximation function *g(t):*

$$g(t) = a_0 + a_1 \cdot \cos(\omega t) + a_2 \cdot \cos(2 \cdot \omega t) + a_3 \cdot \cos(3 \cdot \omega t) + \ldots$$
$$+ b_1 \cdot \sin(\omega t) + b_2 \cdot \sin(2 \cdot \omega t) + b_3 \cdot \sin(3 \cdot \omega t) + \ldots$$

Now, as the next step, the coefficients $a_0, a_1, a_2, a_3, \ldots, b_1, b_2, b_3, \ldots$ are to be determined in a way, that the approximation is an optimum. Therefore, a criterion is necessary to decide, when the approximation is an optimum.

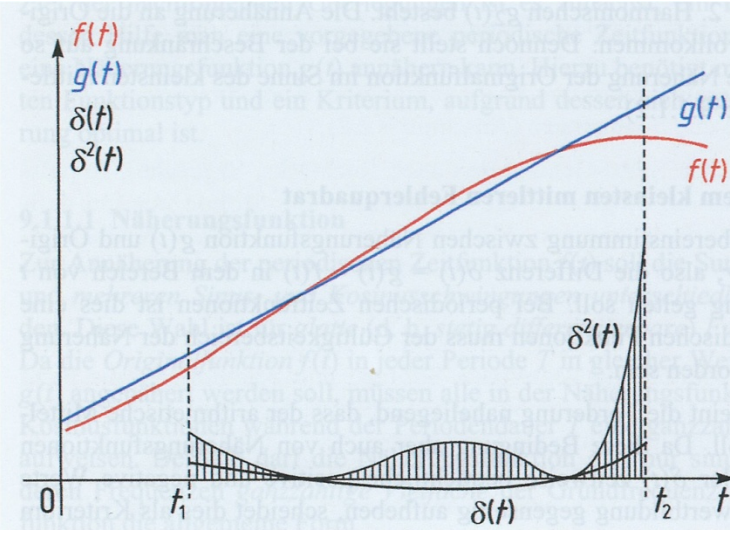
Method of the least error squares

Example:
The function *f(t)* has to be approximated in an optimal way by a straight line given by the equation $g(t) = a_0 + a_1 \cdot t$ in the range $t_1 < t < t_2$.

Error = Difference between approximation function g(t) and original function f(t):

$$\delta(t) = g(t) - f(t)$$

It seems to be obvious, that the arithmetic mean value of the error should be $\overline{\delta} = 0$. However, this is not sufficient: also obviously senseless straight lines with partially enormous positive and negative deviations, which are compensated by the determination of the mean value, fulfill this requirement [1].



Therefore: The average error square should be minimal:

$$\overline{\delta^2} = \frac{1}{t_2 - t_1} \cdot \int_{t_1}^{t_2} [g(t) - f(t)]^2 \, dt$$

With that, the integral has to be minimized, which is shown in the figure above as the shaded area.

$$J = \int_{t_1}^{t_2} [g(t) - f(t)]^2 \, dt = \int_{t_1}^{t_2} [a_0 + a_1 \cdot t - f(t)]^2 \, dt$$

Based on this, the coefficients a_0 and a_1 can be determined. We get them, if we differentiate J with respect to a_0 and a_1, and if we set these first derivatives both to zero:

$$\frac{\partial J}{\partial a_0} = \frac{\partial}{\partial a_0} \int_{t_1}^{t_2} [a_0 + a_1 \cdot t - f(t)]^2 \, dt = 2 \cdot \int_{t_1}^{t_2} [a_0 + a_1 \cdot t - f(t)] dt = 0$$

$$\frac{\partial J}{\partial a_1} = \frac{\partial}{\partial a_1} \int_{t_1}^{t_2} [a_0 + a_1 \cdot t - f(t)]^2 \, dt = 2 \cdot \int_{t_1}^{t_2} [a_0 + a_1 \cdot t - f(t)] \cdot t dt = 0$$

It follows, if we separate both integrals into two partial integrals:

$$\int_{t_1}^{t_2} [a_0 + a_1 \cdot t] dt = a_0 \cdot (t_2 - t_1) + \frac{1}{2} \cdot a_1 \cdot \left(t_2{}^2 - t_1{}^2\right) = \int_{t_1}^{t_2} f(t) dt$$

$$\int_{t_1}^{t_2} [a_0 \cdot t + a_1 \cdot t^2] dt = \frac{1}{2} \cdot a_0 \cdot \left(t_2{}^2 - t_1{}^2\right) + \frac{1}{3} \cdot a_1 \cdot \left(t_2{}^3 - t_1{}^3\right) = \int_{t_1}^{t_2} t \cdot f(t) dt$$

The integrals on the right side have to be solved from the given original function *f(t)* with graphical or numerical methods. The integrals on the left side are solved in a general way, and they give a linear combination of the coefficients a_0 and a_1. A linear system of equations for the determination of the coefficients a_0 and a_1 follows.

The same procedure can be applied, if an approximation by a parabola equation $g(t) = a_0 + a_1 \cdot t + a_2 \cdot t^2$ is used ...

<u>Determination of the Fourier coefficients for a periodic time function</u>

Original function: *f(t)* = periodic time function with angular frequency ω

Approximation function:

$$g(t) = a_0 + a_1 \cdot \cos(\omega t) + a_2 \cdot \cos(2 \cdot \omega t) + a_3 \cdot \cos(3 \cdot \omega t) + ... \\ + b_1 \cdot \sin(\omega t) + b_2 \cdot \sin(2 \cdot \omega t) + b_3 \cdot \sin(3 \cdot \omega t) + ...$$

Following the method of the least error squares the integral J must be minimized:

$$J = \int_{t_0}^{t_0+T} [g(t) - f(t)]^2 \, dt$$

a₀:

$$\frac{\partial J}{\partial a_0} = 2 \cdot \int_{t_0}^{t_0+T} [g(t) - f(t)] dt = 0$$

$$\int_{t_0}^{t_0+T} g(t) dt = a_0 \cdot T = \int_{t_0}^{t_0+T} f(t) dt \qquad \rightarrow \qquad \boxed{a_0 = \frac{1}{T} \cdot \int_{t_0}^{t_0+T} f(t) dt}$$

The integral on the left side has the value $a_0 T$, because the integrals over a full period of all sine and cosine functions contained within $g(t)$ are zero. Therefore the coefficient a_0 represents the direct component.

a_n: $$\frac{\partial J}{\partial a_n} = 2 \cdot \int_{t_0}^{t_0+T} \left[g(t) - f(t) \right] \cdot \cos(n \cdot \omega t) dt = 0$$

$$\int_{t_0}^{t_0+T} g(t) \cdot \cos(n \cdot \omega t) dt = \int_{t_0}^{t_0+T} f(t) \cdot \cos(n \cdot \omega t) dt$$

If the approximation function $g(t)$ is written in the integral on the left side, we can write it as a sum:

$$\int_{t_0}^{t_0+T} a_0 \cdot \cos(n \cdot \omega t) dt + \sum_{x=1}^{m} \int_{t_0}^{t_0+T} a_x \cdot \cos(x \cdot \omega t) \cdot \cos(n \cdot \omega t) dt + \sum_{x=1}^{m} \int_{t_0}^{t_0+T} b_x \cdot \sin(x \cdot \omega t) \cdot \cos(n \cdot \omega t) dt$$

If now the adjacent trigonometry relations are used we can show, that most of the integrals, which are given in the sum, are zero over a full period T. As per definition the integration of a periodic quantity over one or more full periods must always give the value zero.

$$\sin(\alpha) \cdot \sin\beta = \frac{1}{2}\left[\cos(\alpha - \beta) - \cos(\alpha + \beta)\right]$$

$$\cos(\alpha) \cdot \cos\beta = \frac{1}{2}\left[\cos(\alpha - \beta) + \cos(\alpha + \beta)\right]$$

$$\sin(\alpha) \cdot \cos\beta = \frac{1}{2}\left[\sin(\alpha - \beta) + \sin(\alpha + \beta)\right]$$

Only for the integral:

$$\int_{t_0}^{t_0+T} a_x \cdot \cos(x \cdot \omega t) \cdot \cos(n \cdot \omega t) dt$$ for $x = n$ it follows $\cos(x - n) = \cos(0) = 1$:

$$\int_{t_0}^{t_0+T} a_n \cdot \frac{1}{2} \cdot \left[1 + \cos(2 \cdot n \cdot \omega t)\right] dt = \frac{1}{2} \cdot a_n \cdot T$$

Now: $$\frac{1}{2} \cdot a_n \cdot T = \int_{t_0}^{t_0+T} f(t) \cdot \cos(n \cdot \omega t) dt \quad \rightarrow \quad \boxed{a_n = \frac{2}{T} \cdot \int_{t_0}^{t_0+T} f(t) \cdot \cos(n \cdot \omega t) dt}$$

b_n: $$\frac{\partial J}{\partial b_n} = 2 \cdot \int_{t_0}^{t_0+T} \left[g(t) - f(t) \right] \cdot \sin(n \cdot \omega t) dt = 0$$

$$\int_{t_0}^{t_0+T} g(t) \cdot \sin(n \cdot \omega t) dt = \int_{t_0}^{t_0+T} f(t) \cdot \sin(n \cdot \omega t) dt$$

If again the approximation function $g(t)$ is written in the integral on the left side, we can write it as a sum:

$$\int_{t_0}^{t_0+T} a_0 \cdot \sin(n \cdot \omega t)dt + \sum_{x=1}^{m} \int_{t_0}^{t_0+T} a_x \cdot \cos(x \cdot \omega t) \cdot \sin(n \cdot \omega t)dt + \sum_{x=1}^{m} \int_{t_0}^{t_0+T} b_x \cdot \sin(x \cdot \omega t) \cdot \sin(n \cdot \omega t)dt$$

Of course, again all the terms are zero, except:

$$\int_{t_0}^{t_0+T} b_x \cdot \sin(x \cdot \omega t) \cdot \sin(n \cdot \omega t)dt = \int_{t_0}^{t_0+T} b_n \cdot \frac{1}{2} \cdot \left[1 - \cos(2 \cdot n \cdot \omega t)\right]dt = \frac{1}{2} \cdot b_n \cdot T \qquad \text{for } x = n$$

Now: $\frac{1}{2} \cdot b_n \cdot T = \int_{t_0}^{t_0+T} f(t) \cdot \sin(n \cdot \omega t)dt \qquad \rightarrow \qquad \boxed{b_n = \frac{2}{T} \cdot \int_{t_0}^{t_0+T} f(t) \cdot \sin(n \cdot \omega t)dt}$

Generally the agreement of the approximation function *g(t)* with the original function *f(t)* is better, the more terms in the sum are considered. If $n \rightarrow \infty$ the approximation function is identical to the original function.

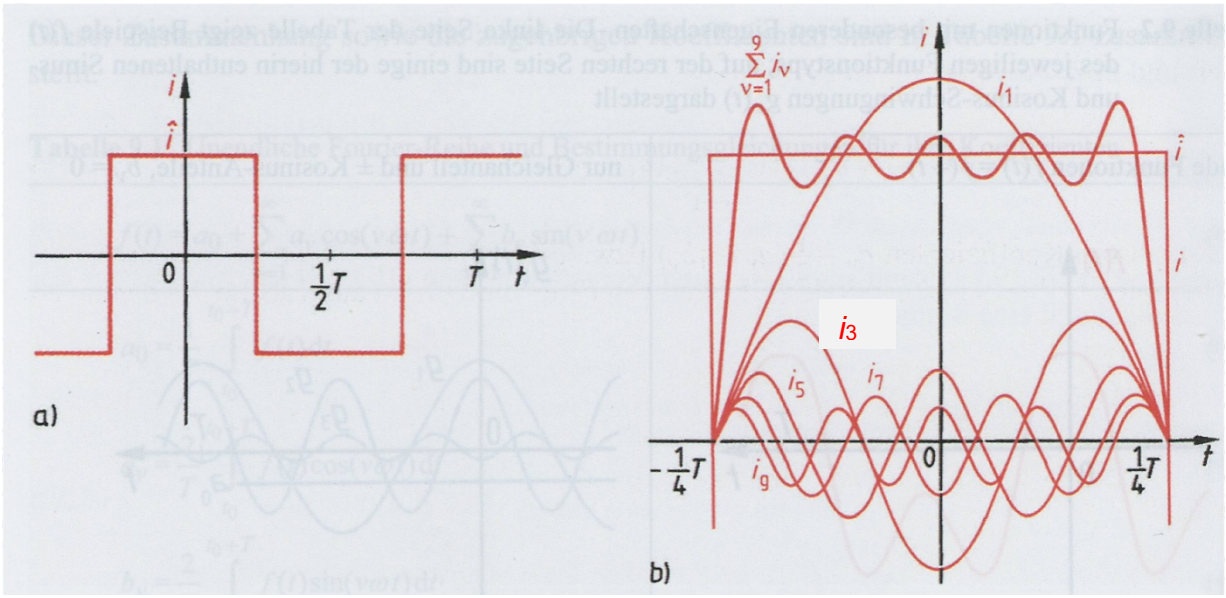
$$\boxed{\begin{aligned} f(t) &= a_0 + \sum_{n=1}^{\infty} a_n \cdot \cos(n \cdot \omega t) + \sum_{n=1}^{\infty} b_n \cdot \sin(n \cdot \omega t) \\ a_0 &= \frac{1}{T} \cdot \int_{t_0}^{t_0+T} f(t)dt \\ a_n &= \frac{2}{T} \cdot \int_{t_0}^{t_0+T} f(t) \cdot \cos(n \cdot \omega t)dt \\ b_n &= \frac{2}{T} \cdot \int_{t_0}^{t_0+T} f(t) \cdot \sin(n \cdot \omega t)dt \end{aligned}}$$

Note: For an **even** function $f(t) = f(-t)$ we have only a_0 and a_n, $b_n = 0$
For an **uneven** function $f(t) = -f(-t)$ we have only b_n, $a_0 = a_n = 0$

Example: Even rectangular current function

The current function is even and alternating [1]. Therefore we get $b_n = 0$ and $a_0 = \frac{1}{T} \cdot \int_{t_0}^{t_0+T} f(t)dt = 0$. For the coefficients a_n it follows:

$$a_n = \frac{2}{T} \cdot \int_{-\frac{1}{4}T}^{\frac{3}{4}T} i(t) \cdot \cos(n \cdot \omega t)dt = \frac{2}{T} \cdot \left\{ \int_{-\frac{1}{4}T}^{\frac{1}{4}T} \hat{i} \cdot \cos(n \cdot \omega t)dt + \int_{\frac{1}{4}T}^{\frac{3}{4}T} -\hat{i} \cdot \cos(n \cdot \omega t)dt \right\}$$



Now we get:

$$a_n = \frac{2 \cdot \hat{i}}{T} \cdot \left\{ \frac{1}{n \cdot \omega} \cdot \left[\sin\left(\frac{n \cdot \omega T}{4} \right) - \sin\left(-\frac{n \cdot \omega T}{4} \right) \right] - \frac{1}{n \cdot \omega} \cdot \left[\sin\left(\frac{3 \cdot n \cdot \omega T}{4} \right) - \sin\left(\frac{n \cdot \omega T}{4} \right) \right] \right\}$$

and with $T = \frac{2\pi}{\omega}$:

$$a_n = \frac{\hat{i}}{n \cdot \pi} \cdot \left\{ \sin\left(\frac{n \cdot \pi}{2} \right) - \sin\left(-\frac{n \cdot \pi}{2} \right) - \sin\left(\frac{3 \cdot n \cdot \pi}{2} \right) + \sin\left(\frac{n \cdot \pi}{2} \right) \right\} = \frac{4 \cdot \hat{i}}{n \cdot \pi} \cdot \sin\left(n \cdot \frac{\pi}{2} \right)$$

With that, all a_n with an even value n are zero. For uneven values n it follows:

$$a_n = \frac{4 \cdot \hat{i}}{n \cdot \pi} = +\frac{1{,}273}{n} \cdot \hat{i} \qquad \text{for } n = 1, 5, 9 \ldots$$

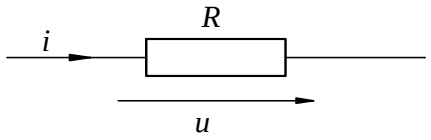
$$a_n = -\frac{4 \cdot \hat{i}}{n \cdot \pi} = -\frac{1{,}273}{n} \cdot \hat{i} \qquad \text{for } n = 3, 7, 11 \ldots$$

For the coefficients up to n = 9 the values are:

a_1	a_3	a_5	a_7	a_9
$+1{,}273 \cdot \hat{i}$	$-0{,}424 \cdot \hat{i}$	$+0{,}255 \cdot \hat{i}$	$-0{,}182 \cdot \hat{i}$	$+0{,}141 \cdot \hat{i}$

5.2 AC voltages applied to basic passive components

Here we investigate finally and generally the outcome of different AC voltages to the three basic passive components of electric networks: resistance, capacitance, inductance.



Resistance R

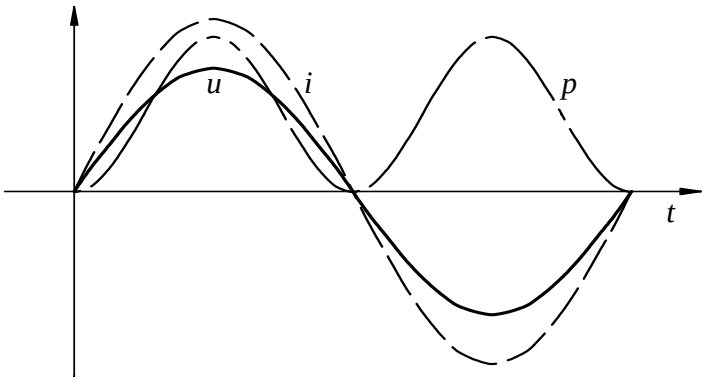
The basic relation is described with (chapter 2):

$$u = R \cdot i$$

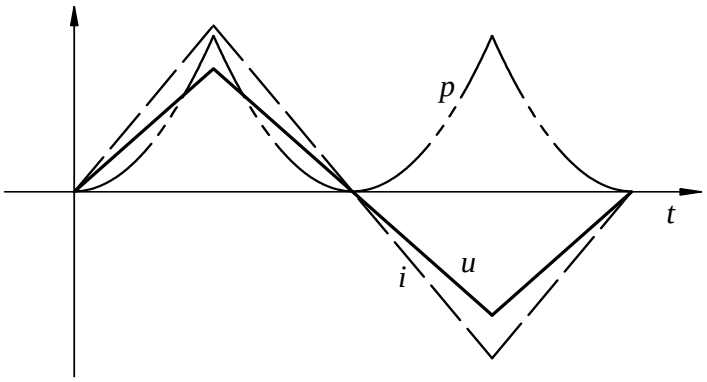
$$i = \frac{1}{R} \cdot u$$

Hence, voltage and current are always in phase and they have exactly the same shape.

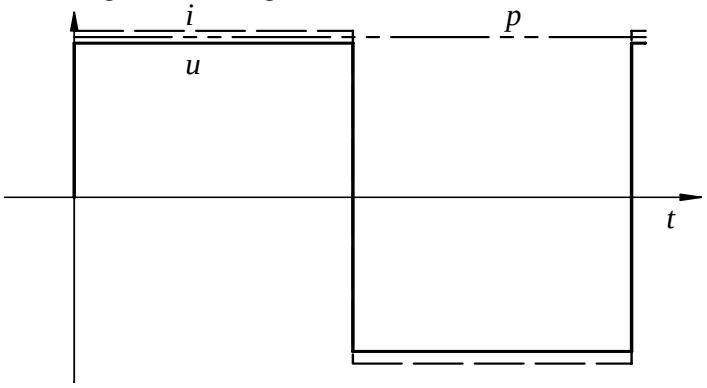
sinusoidal voltage

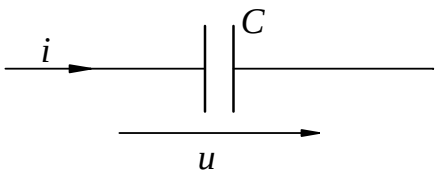


triangular voltage

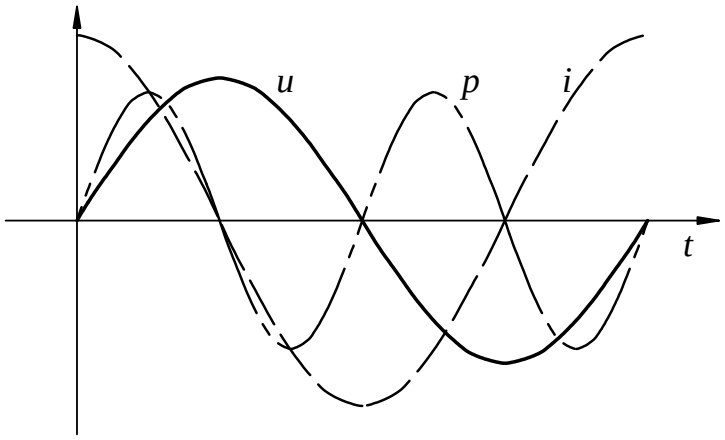


rectangular voltage

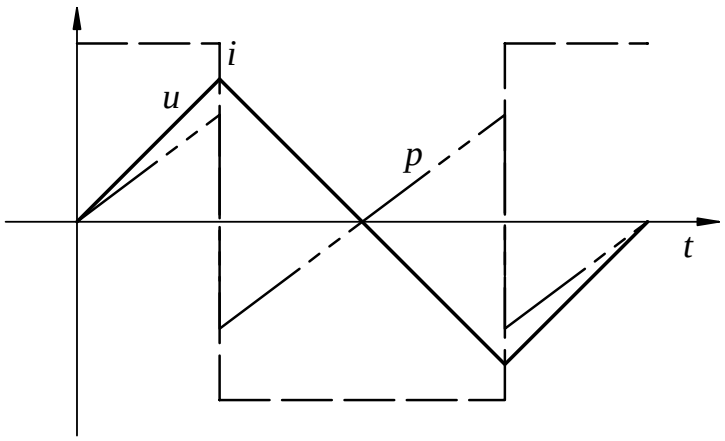




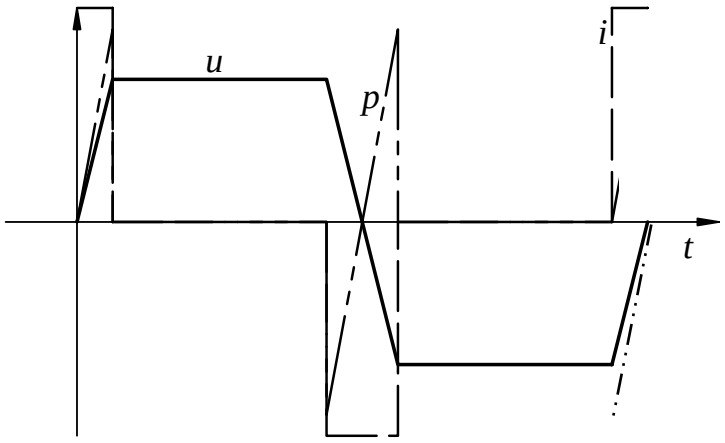
sinusoidal voltage



triangular voltage



rectangular voltage **(trapezoid)**



Capacitance C

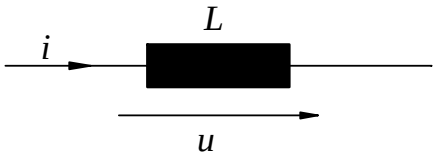
The basic relation is known from chapter 3. It is:

$$u = \frac{1}{C} \cdot \int i \cdot dt$$

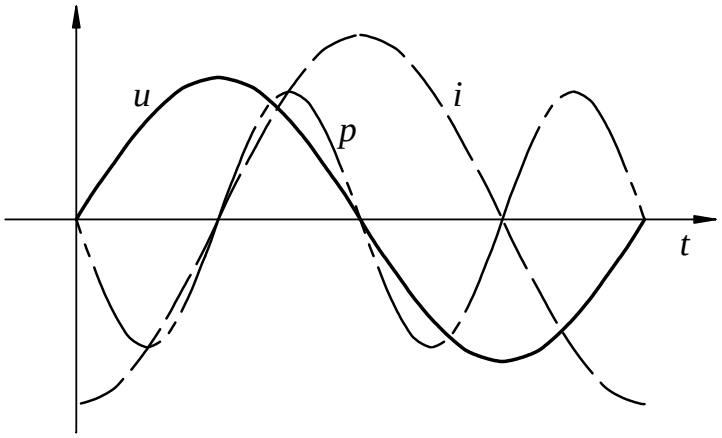
$$i = C \cdot \frac{du}{dt}$$

Knowing the voltage, the current is always proportional to the time derivative of the voltage.

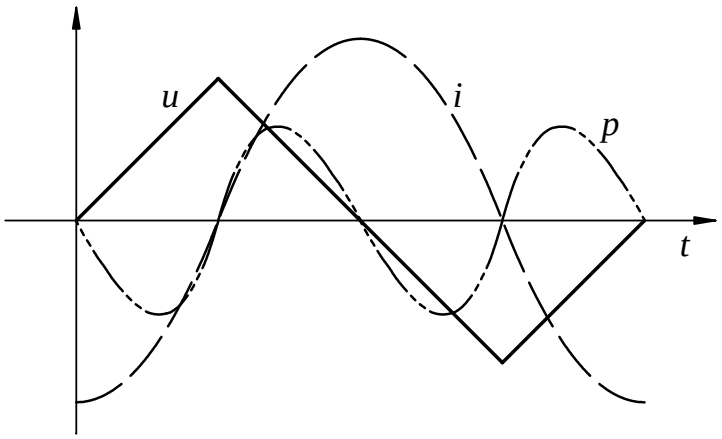
In case of a sinusoidal voltage the phase angle between voltage and current is -90° (the current leads the voltage by 90°).



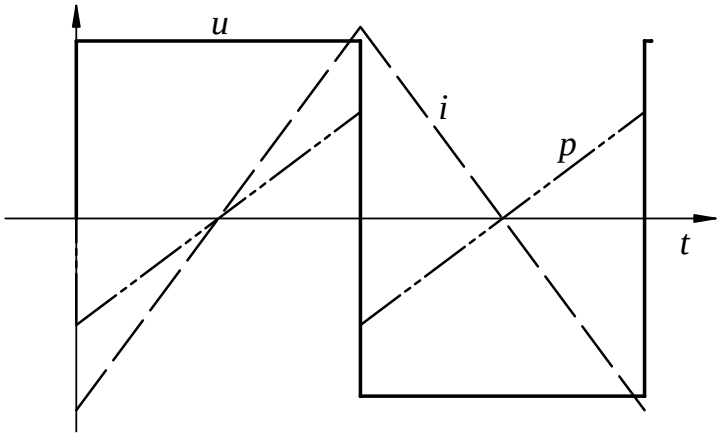
sinusoidal voltage



triangular voltage



rectangular voltage



Inductance L

The basic relation is given by (see chapter 10):

$$u = L \cdot \frac{di}{dt}$$

$$i = \frac{1}{L} \cdot \int u \cdot dt$$

Knowing the voltage, the current is always proportional to the time integral of the voltage.

In case of a sinusoidal voltage the phase angle between voltage and current is +90° (the voltage leads the current by 90°).

5.3 Sinusoidal voltages and currents

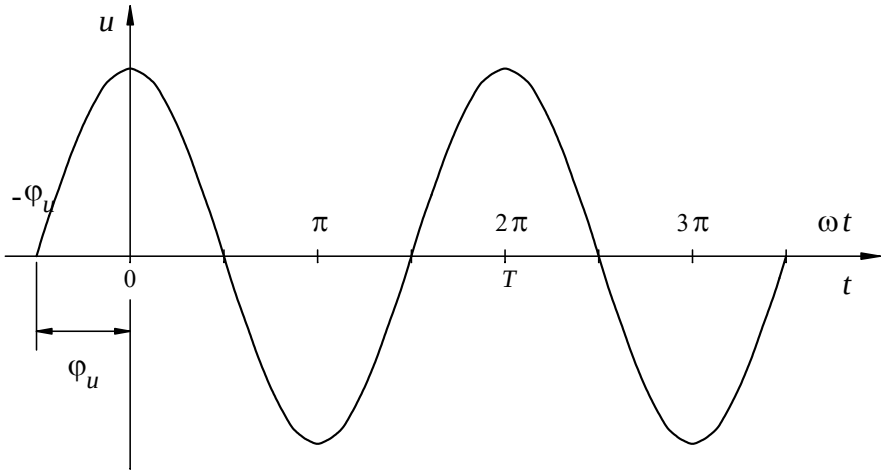
5.3.1 Sinusoidal periodic quantity

Period, frequency, angular frequency and wave length:

$u = \hat{u} \cdot \sin(\omega \cdot t + \varphi_u)$

$\varphi_u > 0 \rightarrow$ leading sine

$\omega \cdot T = 2\pi$



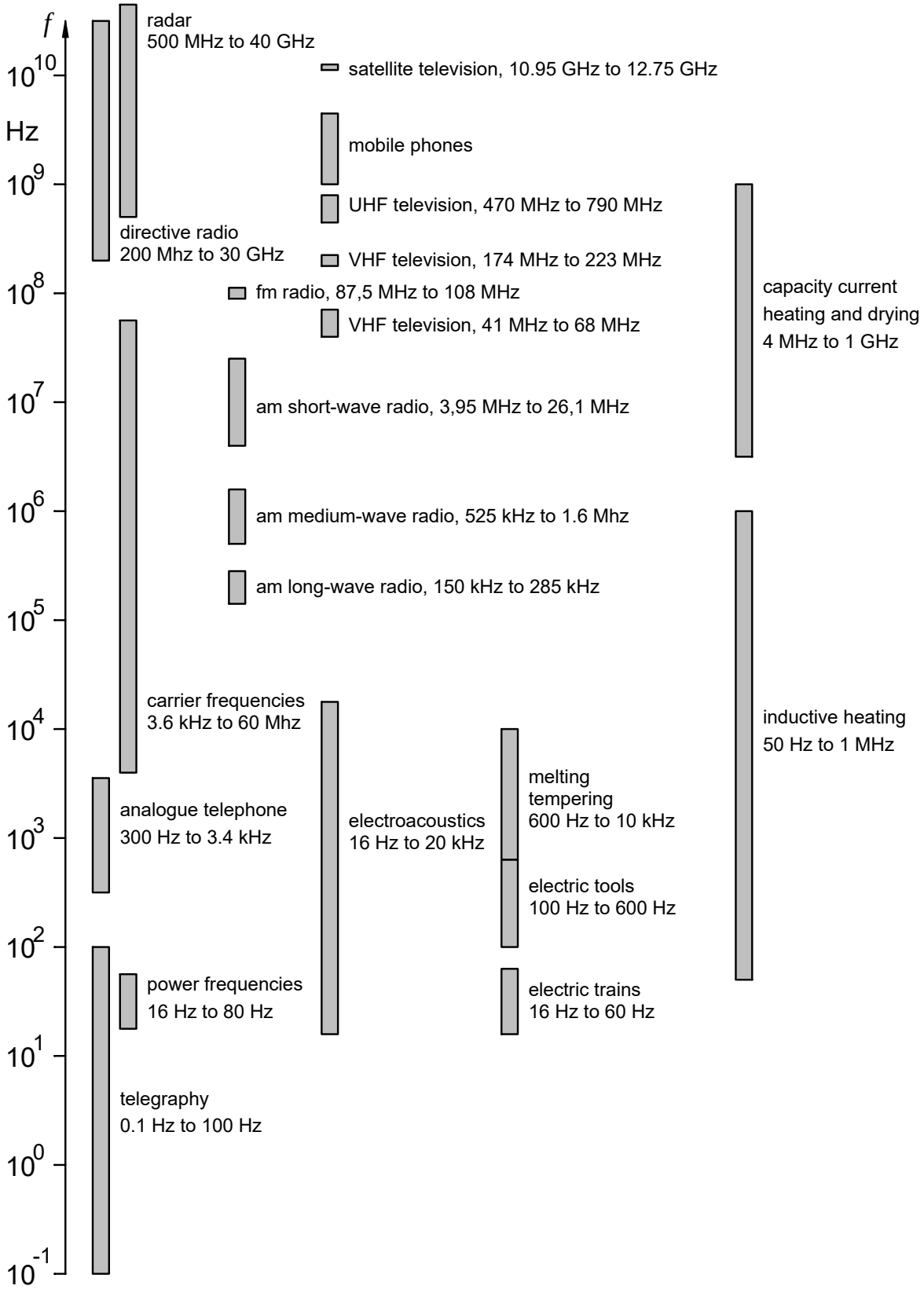
The relations between the quantities used for the description of sinusoidal quantities are given in the following table:

(zero) phase angle	φ	$[\text{rad}, °]$
angular frequency	$\omega = \frac{2 \cdot \pi}{T} = 2 \cdot \pi \cdot f$	$[s^{-1}]$
frequency	$f = \frac{1}{T} = \frac{\omega}{2 \cdot \pi}$	$[\text{Hz}]$
period	$T = \frac{1}{f} = \frac{2 \cdot \pi}{\omega}$	$[s]$
Wave length	$\lambda = c \cdot T = \frac{c}{f}$	$[m]$

The wavelength is the distance covered during one period.

Speed of light $\boxed{c = 3 \cdot 10^8 \frac{m}{s}}$

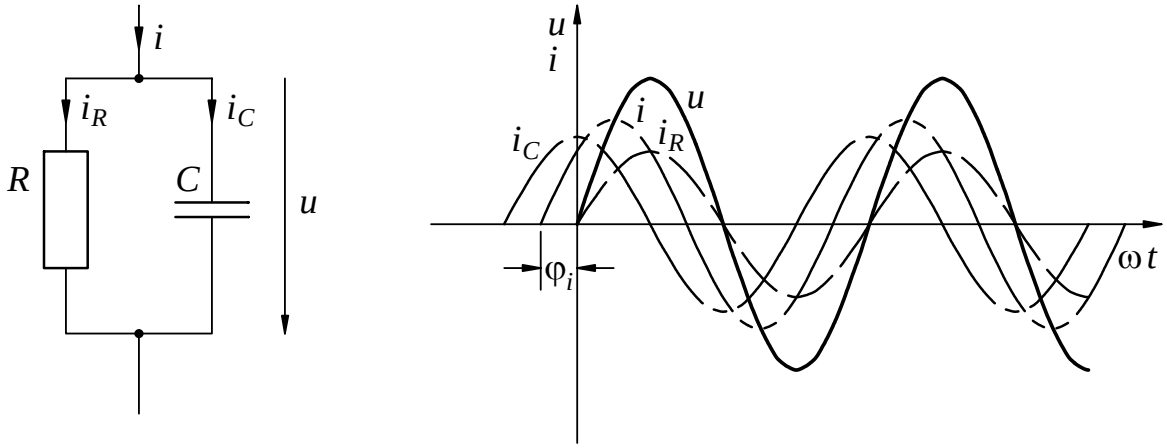
5.3.2 Frequency ranges of technical alternating voltages



Infrared	$3 \cdot 10^{12}...3{,}8 \cdot 10^{14}\,Hz$
Visible light	$3{,}8 \cdot 10^{14}...7{,}9 \cdot 10^{14}\,Hz$ (380...780nm)

5.3.3 Superposition of sinusoidal voltages and currents

Current waveform i(t) for a linear circuit excited by a sinusoidal $u = \hat{u} \cdot \sin(\omega t)$:



Statement: Also the total current *i(t)* as the sum of both partial currents $i_R(t)$ and $i_C(t)$ is sinusoidal and follows the function $i(t) = \hat{i} \cdot \sin(\omega t + \varphi_i)$

Demonstration: $i(t) = \hat{i} \cdot \sin(\omega t + \varphi_i)$

$$\boxed{\sin(\alpha + \beta) = \sin\alpha \cdot \cos\beta + \cos\alpha \cdot \sin\beta}$$

$$i(t) = \hat{i} \cdot \cos\varphi_i \cdot \sin(\omega t) + \hat{i} \cdot \sin\varphi_i \cdot \cos(\omega t)$$

$$i(t) = \hat{i}_R \cdot \sin(\omega t) + \hat{i}_C \cdot \cos(\omega t)$$

$$i(t) = i_R(t) + i_C(t)$$

In a sinusoidally excited linear network, all branch voltages and currents are sinusoids at the same frequency *f* as the excitation signal.

Further demonstration with the example above – calculation of the dissipated power:

$$u = \hat{u} \cdot \sin(\omega t) \qquad i(t) = \hat{i}_R \cdot \sin(\omega t) + \hat{i}_C \cdot \cos(\omega t)$$

$$p(t) = u(t) \cdot i(t) = \hat{u} \cdot \hat{i}_R \cdot \sin^2(\omega t) + \hat{u} \cdot \hat{i}_C \cdot \sin(\omega t) \cdot \cos(\omega t)$$

$$\boxed{\begin{aligned} \sin(\alpha) \cdot \sin\beta &= \frac{1}{2}\left[\cos(\alpha - \beta) - \cos(\alpha + \beta)\right] \\ \sin(\alpha) \cdot \cos\beta &= \frac{1}{2}\left[\sin(\alpha - \beta) + \sin(\alpha + \beta)\right] \end{aligned}}$$

$$p(t) = \frac{\hat{u} \cdot \hat{i}_R}{2} \cdot \left[\cos(\omega t - \omega t) - \cos(\omega t + \omega t)\right] + \frac{\hat{u} \cdot \hat{i}_C}{2} \cdot \left[\sin(\omega t - \omega t) + \sin(\omega t + \omega t)\right]$$

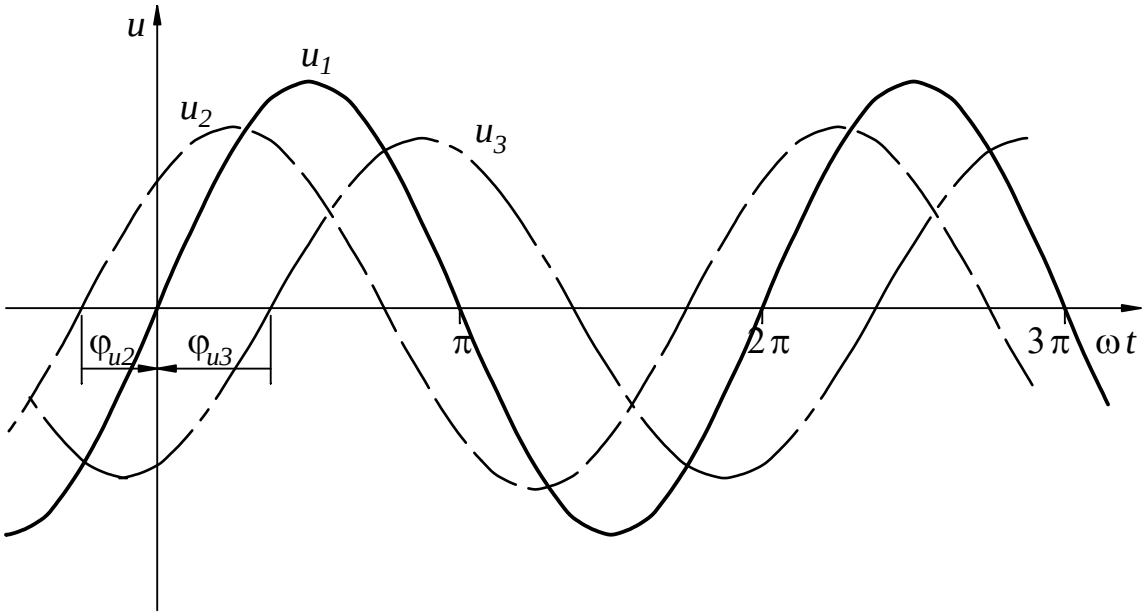
$$p(t) = \frac{\hat{u} \cdot \hat{i}_R}{2} \cdot \left[1 - \cos(2 \cdot \omega t)\right] + \frac{\hat{u} \cdot \hat{i}_C}{2} \cdot \sin(2 \cdot \omega t)$$

$$P = \overline{p(t)} = \frac{\hat{u} \cdot \hat{i}_R}{2} = \frac{\hat{u}}{\sqrt{2}} \cdot \frac{\hat{i}_R}{\sqrt{2}} = U \cdot I_R$$

The power dissipated by any circuit element has an average component plus a sinusoidal component oscillating at a frequency double that of the original source frequency.

5.4 Representation of sinusoidal alternating quantities

5.4.1 The time function



Time functions of three sinusoidal voltages:

$$u_1 = \hat{u}_1 \cdot \sin(\omega t)$$

$$u_2 = \hat{u}_2 \cdot \sin(\omega t + \varphi_{u2})$$ with $\varphi_{u2} > 0$, leading u_1 !

$$u_3 = \hat{u}_3 \cdot \sin(\omega t + \varphi_{u3})$$ with $\varphi_{u3} < 0$, lagging u_1 !

To determine the zero phase angle φj, the positive zero crossing of the function closest to the origin is regarded. The zero phase angle is marked by a (reference) arrow in direction to the origin. The sign of the zero phase angle is defined like shown in the example of the figure above.

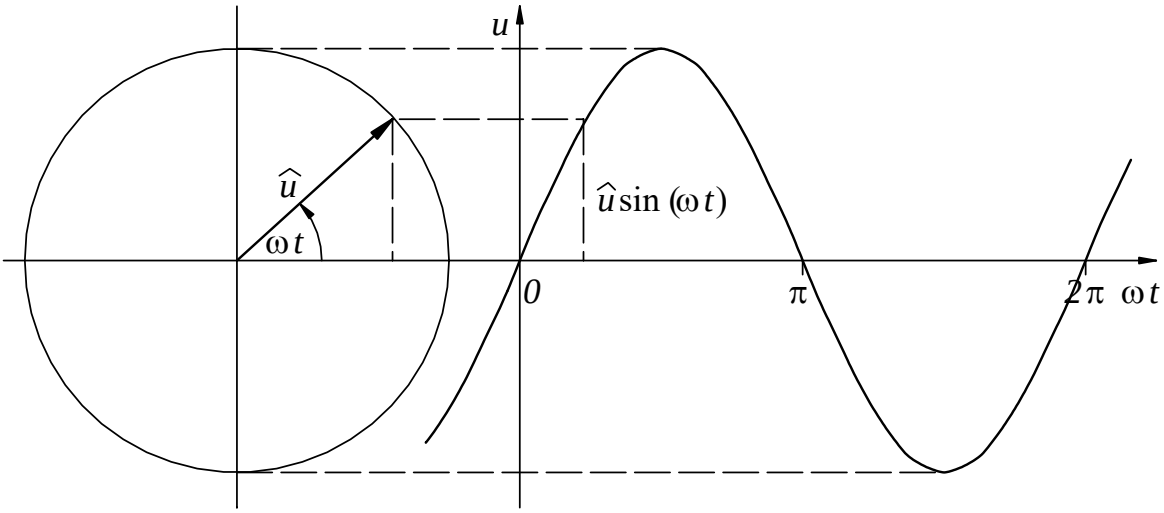
Zero phase angles of ±90°:

$$\varphi = 90° \quad \sin(\omega t + 90°) = \sin\left(\omega t + \frac{\pi}{2}\right) = \cos(\omega t) \qquad \text{leading}$$

$$\varphi = -90° \quad \sin(\omega t - 90°) = \sin\left(\omega t - \frac{\pi}{2}\right) = -\cos(\omega t) \qquad \text{lagging}$$

5.4.2 Peak value phasor diagram

A sinusoid can be described by a rotating phasor with the length corresponding to the peak value of the sinusoid.



Statement: The sum of two sinusoidal voltages (currents) of frequency *f* is a sinusoidal voltage (current) of the same frequency *f* (see next figure).

If this is true, we have to determine the peak value $\hat{u}$ and the zero phase angle φ_u of the resulting voltage:

$$\hat{u}_1 \cdot \sin(\omega t) + \hat{u}_2 \cdot \sin(\omega t + \varphi_{u2}) = \hat{u} \cdot \sin(\omega t + \varphi_u)$$

With $\boxed{\sin(\alpha + \beta) = \sin\alpha \cdot \cos\beta + \cos\alpha \cdot \sin\beta}$ we get:

$$\hat{u}_1 \cdot \sin(\omega t) + \hat{u}_2 \cdot [\sin(\omega t) \cdot \cos\varphi_{u2} + \cos(\omega t) \cdot \sin\varphi_{u2}] = \hat{u} \cdot [\sin(\omega t)\cos\varphi_u + \cos(\omega t) \cdot \sin\varphi_u]$$

$$(\hat{u}_1 + \hat{u}_2 \cdot \cos\varphi_{u2} - \hat{u} \cdot \cos\varphi_u) \cdot \sin(\omega t) + (\hat{u}_2 \cdot \sin\varphi_{u2} - \hat{u} \cdot \sin\varphi_u) \cdot \cos(\omega t) = 0$$

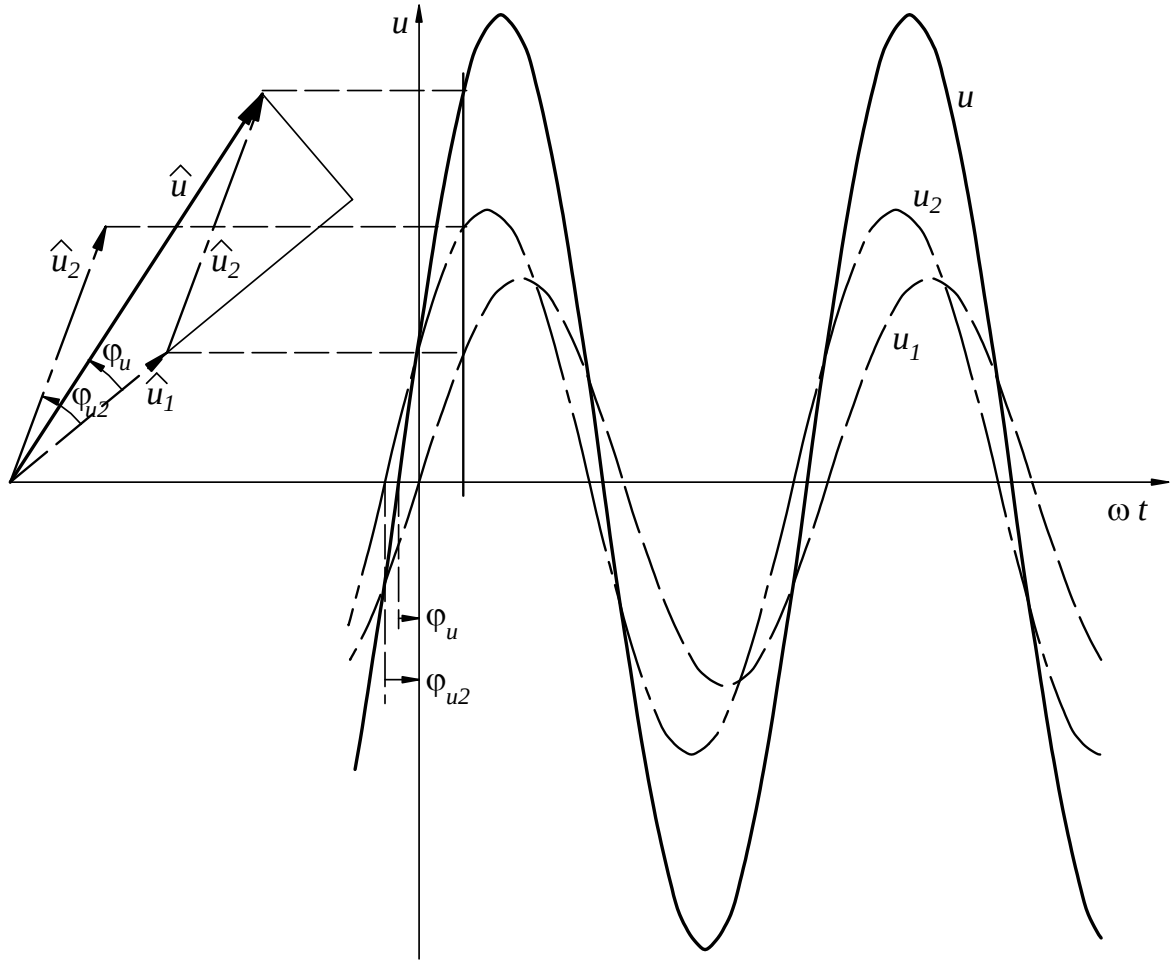
This last equation is valid for any value of *t* only, if the two expressions in parenthesis are equal to zero:

$$\hat{u}_1 + \hat{u}_2 \cdot \cos\varphi_{u2} = \hat{u} \cdot \cos\varphi_u \qquad (1)$$

$$\hat{u}_2 \cdot \sin \varphi_{u2} = \hat{u} \cdot \sin \varphi_u \qquad (2)$$

If we compare these equations with the phasor diagram above, we see:

Phasors are added geometrically (like vectors)



Equation (1) and (2) represent the components of the resulting phasor *u*. To determine the peak value $\hat{u}$ we have to square both equations and add them:

$$\hat{u}_1^{\,2} + 2 \cdot \hat{u}_1 \cdot \hat{u}_2 \cdot \cos \varphi_{u2} + \hat{u}_2^{\,2} \cdot \cos^2 \varphi_{u2} + \hat{u}_2^{\,2} \cdot \sin^2 \varphi_{u2} = \hat{u}^2 \cdot \cos^2 \varphi_u + \hat{u}^2 \cdot \sin^2 \varphi_u$$

$$\hat{u}_1^{\,2} + 2 \cdot \hat{u}_1 \cdot \hat{u}_2 \cdot \cos \varphi_{u2} + \hat{u}_2^{\,2} = \hat{u}^2$$

$$\hat{u} = \sqrt{\hat{u}_1^{\,2} + 2 \cdot \hat{u}_1 \cdot \hat{u}_2 \cdot \cos \varphi_{u2} + \hat{u}_2^{\,2}}$$

The tangens of the zero phase angle is equal to the ratio of the two components:

$$\tan \varphi_u = \frac{\hat{u}_2 \cdot \sin \varphi_{u2}}{\hat{u}_1 + \hat{u}_2 \cdot \cos \varphi_{u2}}$$

The fact, that we get unique results for $\hat{u}$ and φ_u, that do not depend on the time *t* shows, that the statement made at the beginning, is correct.

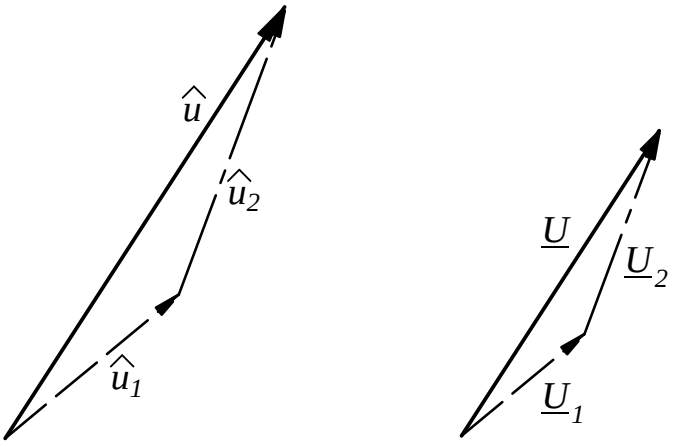
5.4.3 RMS phasor diagram

Instead of using the peak values a phasor diagram can also be drawn using the RMS values of the sinusoids. The main differences between the two diagrams are

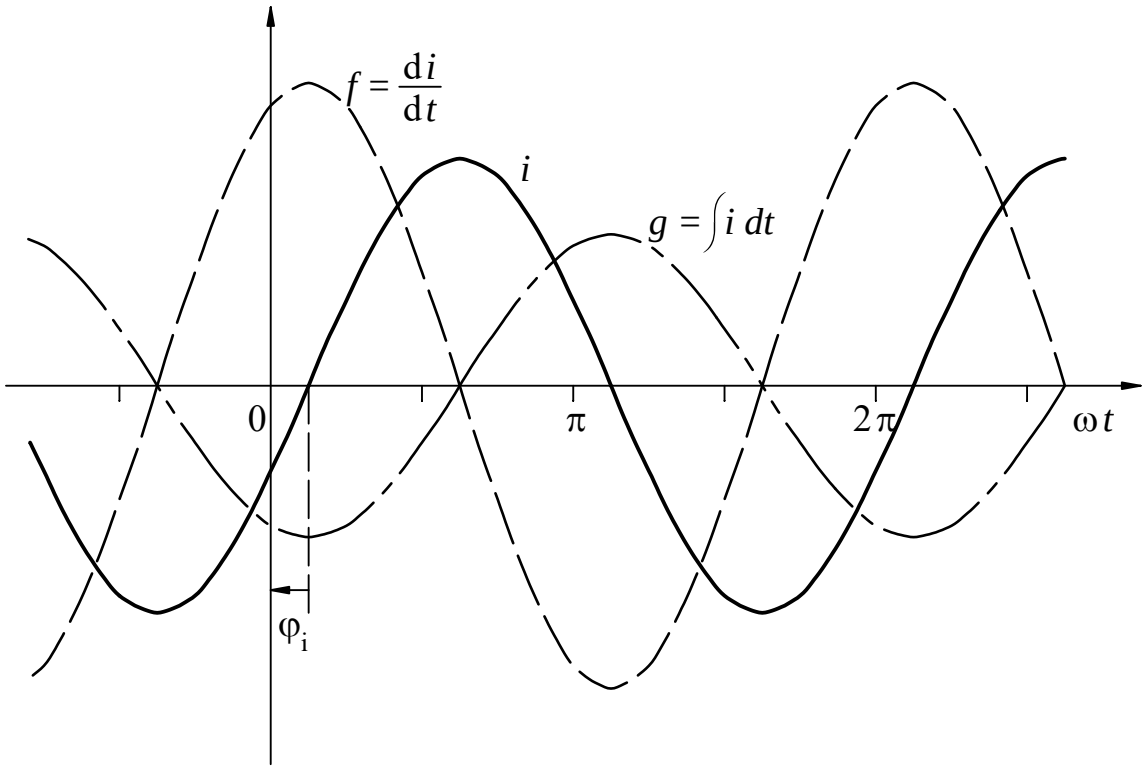
- The length of a peak value phasor corresponds to the peak value of the quantity.
- The length of an RMS phasor corresponds to the RMS value of the quantity. For a sinusoid it is shorter by a factor of $1/\sqrt{2}$ compared to the peak value.
- The peak value phasors are regarded as rotating with the angular frequency ω.
- The RMS phasors are regarded as non moving.

The latter difference is simply a convention. Phasors are added geometrically (like vectors):

- The position of the first phasor is arbitrary.
- The orientation of the phasors relative to each other is uniquely defined.



5.4.4 Differentiation and integration of sinusoidal waveforms



We analyse the time function of a sinusoid current, its derivative and its integral. The time function shown is given by:

$$i(t) = \hat{i} \cdot \sin(\omega t + \varphi_i)$$

It‘s zero phase angle $\varphi_i < 0$; the sinusoid is lagging.

Differentiation: $$f = \frac{di}{dt} = \omega \cdot \hat{i} \cdot \cos(\omega t + \varphi_i)$$

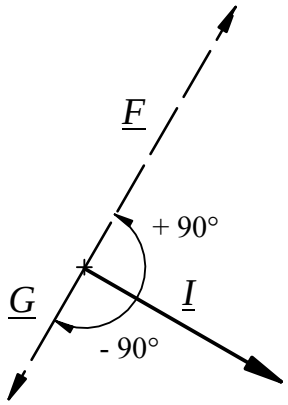
$$f = \omega \cdot \hat{i} \cdot \sin\left(\omega t + \varphi_i + \frac{\pi}{2}\right)$$

Integration: $$g = \int i \cdot dt = \frac{1}{\omega} \cdot \hat{i} \cdot [-\cos(\omega t + \varphi_i)]$$

$$g = \frac{\hat{i}}{\omega} \cdot \sin\left(\omega t + \varphi_i - \frac{\pi}{2}\right)$$

Differentiation	→	multiplication by ω and rotation by + 90°
Integration	→	division by ω and rotation by – 90°

All three functions now can be shown in the RMS phasor diagram



5.4.5 RMS phasor diagram in the complex Gaussian plane

Phasors are two dimensional vectors that can be described by two components. For many mathematical operations it is convenient to represent a phasor by a complex number. To avoid confusion with the instantaneous value of the current *i,* we use for complex AC calculations the symbol *j* for the imaginary unit.

A complex number $\underline{r}$ has a real part *a* and an imaginary part *b* and can be written as:

$$\underline{r} = a + jb$$

Definition: $j = \sqrt{-1}$ and $j^2 = -1$

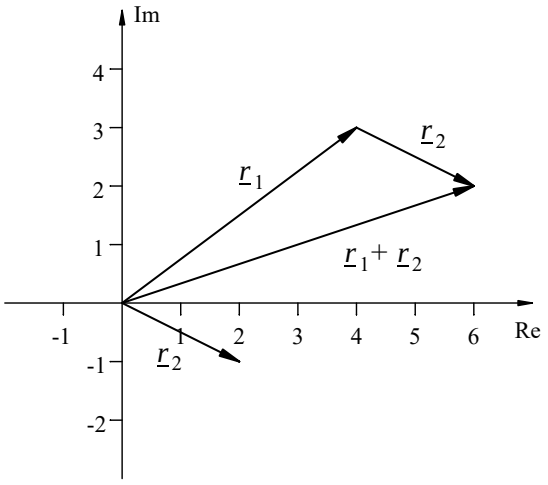
An important consequence of this definition is, that: $\frac{1}{j} = \frac{j}{j \cdot j} = \frac{j}{-1} = -j$

The addition of two complex numbers is performed by adding their real parts andimaginary parts respectively.

Example (see figure):

$$\underline{r}_1 = 4 + j3$$
$$\underline{r}_2 = 2 - j$$

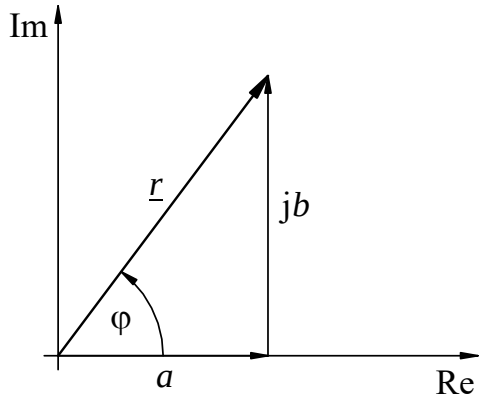
$$\underline{r} = \underline{r}_1 + \underline{r}_2 = 4 + j3 + 2 - j = \underline{\underline{6 + j2}}$$



Multiplication of two complex numbers:

$$\underline{r} = \underline{r}_1 \cdot \underline{r}_2 = (4 + j3) \cdot (2 - j)$$
$$= 8 + j6 - j4 - 3j^2 = \underline{\underline{11 + j2}}$$

As shown above, a phasor in the complex plane can be described by its real part *a* and its imaginary part *b*. Another possibility is to give its magnitude or length *r* and the angle between the phasor and the real axis φ.



Representation of complex numbers	
cartesian form rectangular coordinates	exponential form polar coordinates
$\underline{r} = a + jb$	$\underline{r} = re^{j\varphi}$
conversion formulae	
$a = r\cos\varphi$	$r = \sqrt{a^2 + b^2}$
$b = r\sin\varphi$	$\varphi = \arctan\frac{b}{a}$

The exponential form can be derived from the Eulerian equation:

$$\boxed{\cos(\varphi) + j \cdot \sin(\varphi) = e^{j\varphi}}$$

Addition and subtraction of complex numbers can only be done if the numbers aregiven in cartesian form (rectangular coordinates).

Multiplication and **division** of complex numbers can be performed most easily, if the numbers are given in the exponential form (polar coordinates):

$$\underline{r}_1 \cdot \underline{r}_2 = \left(r_1 \cdot e^{j\varphi_1}\right) \cdot \left(r_2 \cdot e^{j\varphi_2}\right) = r_1 \cdot r_2 \cdot e^{j(\varphi_1 + \varphi_2)}$$

$$\underline{r}_1 \div \underline{r}_2 = \frac{\left(r_1 \cdot e^{j\varphi_1}\right)}{\left(r_2 \cdot e^{j\varphi_2}\right)} = \frac{r_1}{r_2} \cdot e^{j(\varphi_1 - \varphi_2)}$$

However, these operations can also be done in cartesian form:

$$\underline{r}_1 \cdot \underline{r}_2 = (a_1 + jb_1) \cdot (a_2 + jb_2) = a_1 \cdot a_2 + ja_1 \cdot b_2 + ja_2 \cdot b_1 - b_1 \cdot b_2$$
$$\underline{r}_1 \cdot \underline{r}_2 = (a_1 \cdot a_2 - b_1 \cdot b_2) + j(a_1 \cdot b_2 + a_2 \cdot b_1)$$

$$\frac{\underline{r}_1}{\underline{r}_2} = \frac{a_1 + jb_1}{a_2 + jb_2} = \frac{(a_1 + jb_1) \cdot (a_2 - jb_2)}{(a_2 + jb_2) \cdot (a_2 - jb_2)} = \frac{a_1 \cdot a_2 - ja_1 \cdot b_2 + ja_2 \cdot b_1 + b_1 \cdot b_2}{a_2^{\,2} - ja_2 \cdot b_2 + ja_2 \cdot b_2 + b_2^{\,2}}$$
$$\frac{\underline{r}_1}{\underline{r}_2} = \frac{a_1 \cdot a_2 + b_1 \cdot b_2}{a_2^{\,2} + b_2^{\,2}} + j\frac{a_2 \cdot b_1 - a_1 \cdot b_2}{a_2^{\,2} + b_2^{\,2}}$$

Some special complex numbers of magnitude one:

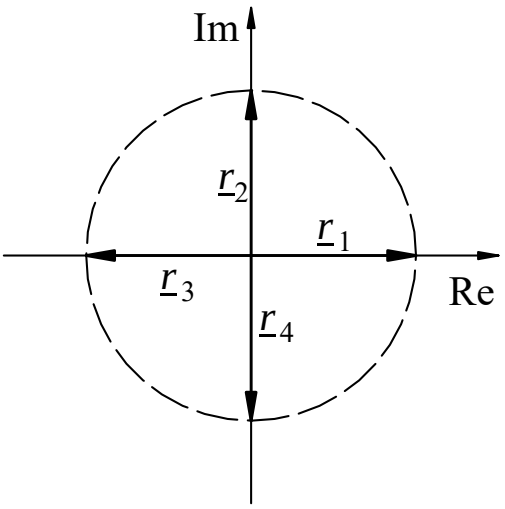
$$\underline{r}_1 = 1 \cdot e^{j0} = e^{j0°} = \cos 0° + j \sin 0° = +1$$

$$\underline{r}_2 = 1 \cdot e^{j\frac{\pi}{2}} = e^{j90°} = \cos 90° + j \sin 90° = +j$$

$$\underline{r}_3 = 1 \cdot e^{j\pi} = e^{j180°} = \cos 180° + j \sin 180° = -1$$

$$\underline{r}_4 = 1 \cdot e^{j\frac{3\pi}{2}} = e^{j270°} = e^{-j\frac{\pi}{2}} = e^{-j90°} =$$

$$= \cos(-90°) + j \sin(-90°) = -j = \frac{1}{j}$$



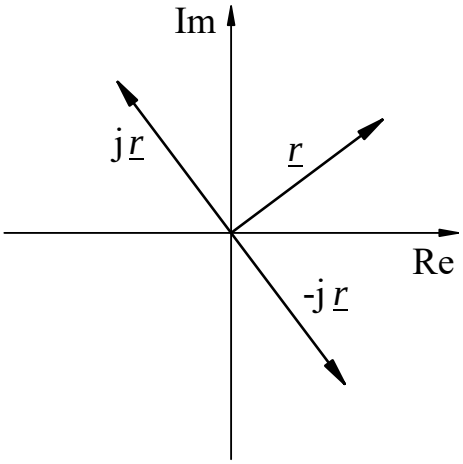
Multiplication of an arbitrary complex number $\underline{r}$ by j or by - j:

$$j \cdot \underline{r} = e^{j90°} \cdot r \cdot e^{j\varphi} = r \cdot e^{j(\varphi+90°)}$$

rotation of the phasor r by **+ 90°**
(like required for the differentiation)

$$\frac{\underline{r}}{j} = -j \cdot \underline{r} = e^{-j90°} \cdot r \cdot e^{j\varphi} = r \cdot e^{j(\varphi-90°)}$$

rotation of the phasor r by **– 90°**
(like required for the integration)



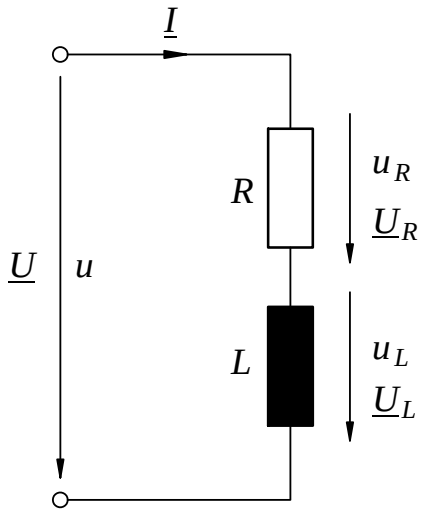
5.5 Design and analysis of AC circuits

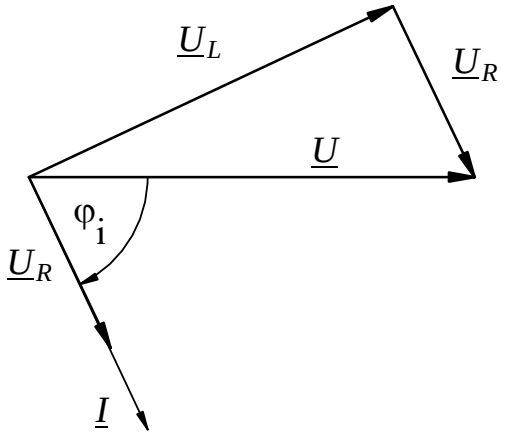
5.5.1 Basic considerations

The following table shows the different ways to perform operations on sinusoidals of the same frequency:

	time functions	phasor diagrams	complex quantities
Addition	superposition of the time functions	geometrical addition of phasors	addition of the complex quantities
Multiplication with (1-)	sign shift, i.e.phase shift by 180°	phasor reversal, rotation of phasor by 180°	multiplication of the complex quantity by (-1)
multiplication by a positive real factor			
Differentiation	differentiation, i.e. phase shift by + 90° and multiplication by ω	rotation by + 90° and multiplication by ω	multiplication of the complex quantity by $j\omega$
Integration	integration, i.e. phase shift by - 90° and division by ω	rotation by - 90° and division by ω	division of the complex quantity by $j\omega$

Example for an AC - circuit and corresponding phasor diagram (the phasor for $\underline{U}$ is arbitrarily positioned along the real axis)





The lower case letters represent quantities varying with time while the underlined upper case letters stand for complex RMS values.

For the time functions we get:

$$u = u_R + u_L = R \cdot i + L \cdot \frac{di}{dt}$$

If we express the same relation in complex RMS values, we get:

$$\underline{U} = \underline{U}_R + \underline{U}_L = R \cdot \underline{I} + j\omega L \cdot \underline{I}$$

$$\underline{U} = \left(R + j\omega L\right) \cdot \underline{I} \qquad \text{and} \qquad \underline{I} = \frac{\underline{U}}{R + j\omega L}$$

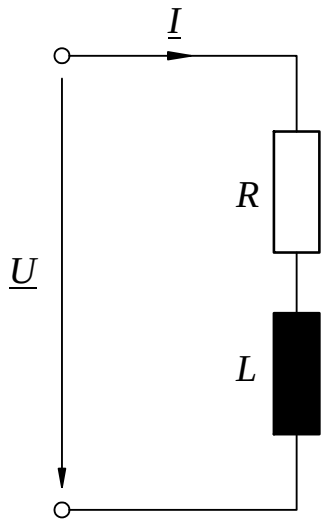
We convert the last equation to polar coordinates:

$$\underline{I} = \frac{U \cdot e^{j0}}{\sqrt{R^2 + \left(\omega L\right)^2} \cdot e^{j \arctan \frac{\omega L}{R}}} = \frac{U}{\sqrt{R^2 + \left(\omega L\right)^2}} e^{-j \arctan \frac{\omega L}{R}}$$

$$I = \frac{U}{\sqrt{R^2 + \left(\omega L\right)^2}}; \qquad \varphi_i = -\arctan \frac{\omega L}{R}$$

5.5.2 Complex resistance and conductance

Example:



The relation between current and voltage is given by:

$$\underline{U} = \left(R + j\omega L\right) \cdot \underline{I}$$

The term in parenthesis is the complex resistance $\underline{Z}$:

$$\underline{U} = \underline{Z} \cdot \underline{I}$$
$$\underline{Z} = R + jX = Z \cdot e^{j\varphi}$$

The reciprocal of the equations above is:

$$\underline{I} = \underline{Y} \cdot \underline{U}$$
$$\underline{Y} = G + jB = Y \cdot e^{j\varphi_y}$$

Definitions:

	$\underline{Z}$	= complex resistance, impedance	= resistance operator
R =	Re $\underline{Z}$	= resistance	
X =	Im $\underline{Z}$	= reactance	
Z =	$\lvert\underline{Z}\rvert$	= impedance	
φ =	Arg $\underline{Z}$	= phase angle (= φ_z)	
	$\underline{Y}$	= complex conductance, admittance	= conductance operator
G =	Re $\underline{Y}$	= conductance	
B =	Im $\underline{Y}$	= susceptance	
Y =	$\lvert\underline{Y}\rvert$	= admittance	
φ_y =	Arg $\underline{Y}$	= phase angle	

In polar coordinates the relation between $\underline{Z}$ and $\underline{Y}$ is:

$$\underline{Y} = \frac{1}{\underline{Z}} = \frac{1}{Z \cdot e^{j\varphi}} = \frac{1}{Z} \cdot e^{-j\varphi} = Y \cdot e^{j\varphi_y} \qquad \text{also:} \quad \boxed{Y = \frac{1}{Z} \qquad \varphi_y = -\varphi}$$

In rectangular coordinates we get:

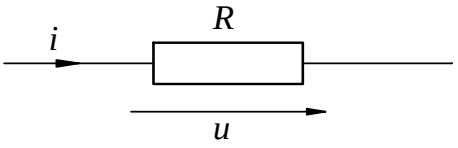
$$\underline{Y} = \frac{1}{\underline{Z}} = \frac{1}{R + jX} = \frac{1}{R + jX} \cdot \frac{R - jX}{R - jX} = \frac{R - jX}{R^2 + X^2} = \frac{R}{R^2 + X^2} + j\frac{-X}{R^2 + X^2} = G + jB$$

with that: $\boxed{G = \frac{R}{R^2 + X^2} \qquad B = \frac{-X}{R^2 + X^2}}$

5.5.3 Basic components *R*, *L* and *C*

excited by a current:

$$i(t) = \hat{i} \cdot \sin(\omega t + \varphi_i)$$

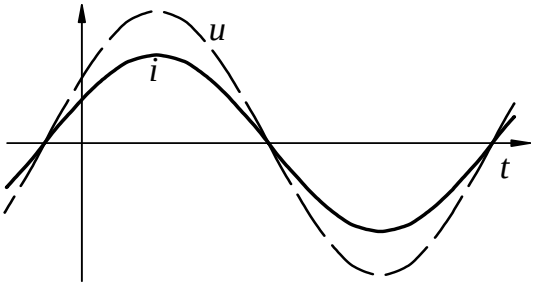


Ohmic resistance

$$u = R \cdot i$$

time function

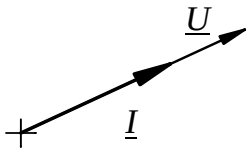
$$u(t) = R \cdot \hat{i} \cdot \sin(\omega t + \varphi_i)$$



voltage/current phasor diagram

$$\underline{U} = R \cdot \underline{I}$$

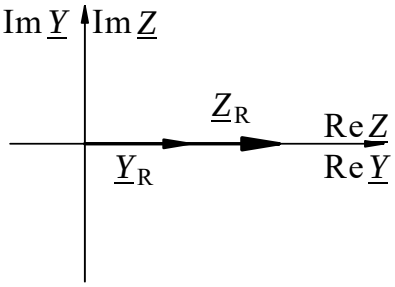
$$\underline{I} = \frac{1}{R} \cdot \underline{U}$$



impedance/admittance phasor diagram

$$\underline{Z}_R = R$$

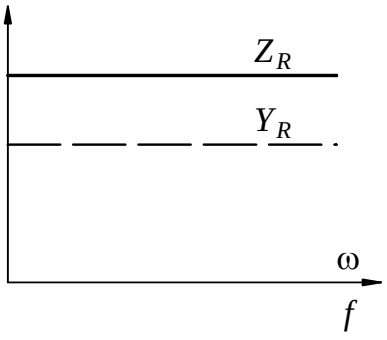
$$\underline{Y}_R = \frac{1}{R}$$



magnitude of impedance and admittance as a function of (angular) frequency

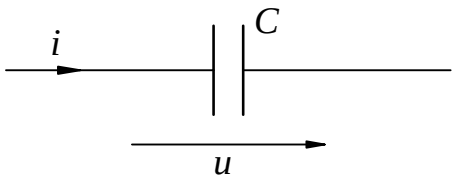
$$Z_R = R$$

$$Y_R = \frac{1}{R}$$



Capacitance

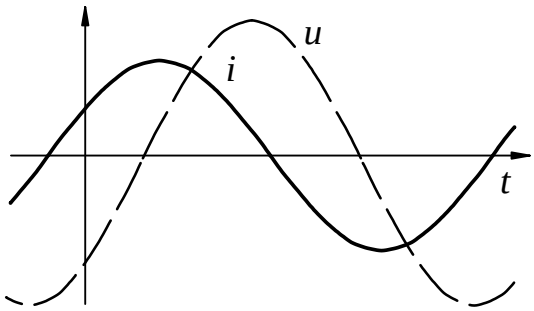
$$u = \frac{1}{C} \cdot \int i dt$$



time function

$$u(t) = \frac{1}{\omega C} \cdot \hat{i} \cdot \left[-\cos\left(\omega t + \varphi_i\right) \right]$$

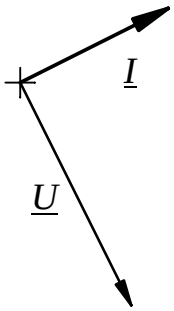
$$u(t) = \frac{1}{\omega C} \cdot \hat{i} \cdot \sin\left(\omega t + \varphi_i - 90°\right)$$



voltage/current phasor diagram

$$\underline{U} = \frac{1}{j\omega C} \cdot \underline{I}$$

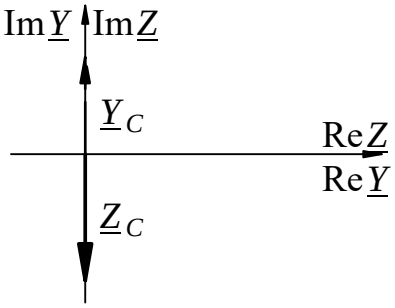
$$\underline{I} = j\omega C \cdot \underline{U}$$



impedance/admittance phasor diagram

$$\underline{Z}_C = \frac{1}{j\omega C}$$

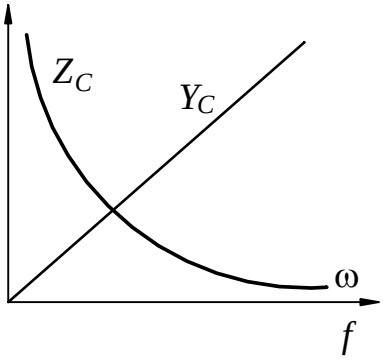
$$\underline{Y}_C = j\omega C$$



magnitude of impedance and admittance as a function of (angular) frequency

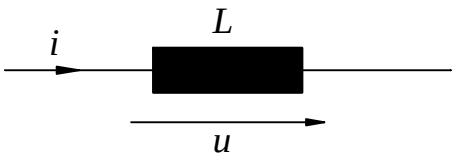
$$Z_C = \frac{1}{\omega C}$$

$$Y_C = \omega C$$



Inductance

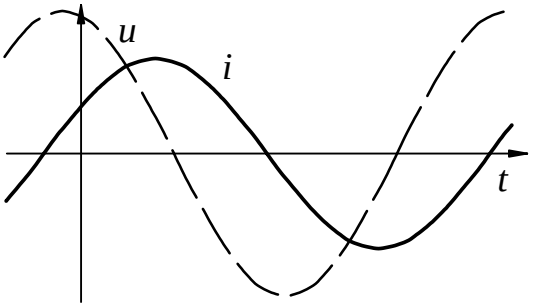
$$u = L \cdot \frac{di}{dt}$$



time function

$$u(t) = \omega L \cdot \hat{i} \cdot \cos(\omega t + \varphi_i)$$

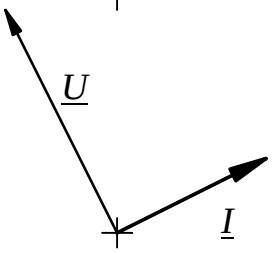
$$u(t) = \omega L \cdot \hat{i} \cdot \sin(\omega t + \varphi_i + 90°)$$



voltage/current phasor diagram

$$\underline{U} = j\omega L \cdot \underline{I}$$

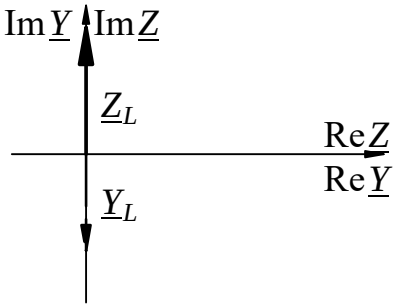
$$\underline{I} = \frac{1}{j\omega L} \cdot \underline{U}$$



impedance/admittance phasor diagram

$$\underline{Z}_L = j\omega L$$

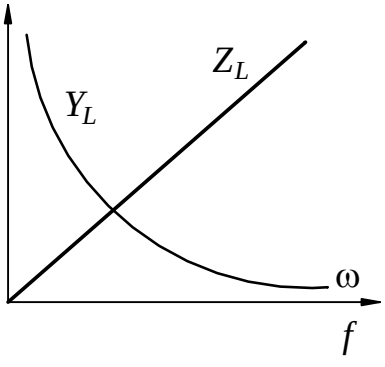
$$\underline{Y}_L = \frac{1}{j\omega L}$$



magnitude of impedance and admittance as a function of (angular) frequency

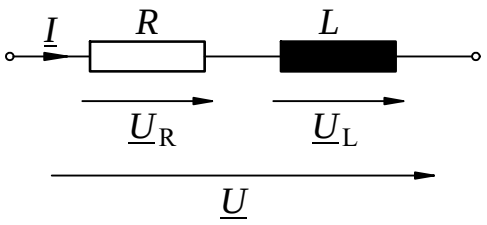
$$Z_L = \omega L$$

$$Y_L = \frac{1}{\omega L}$$



5.5.4 RL series circuit

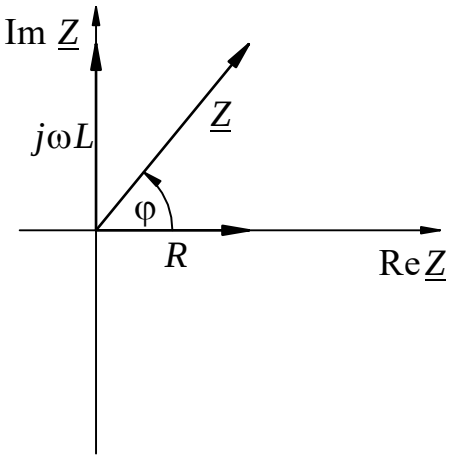
$$\underline{U} = \underline{U}_R + \underline{U}_L = R \cdot \underline{I} + j\omega L \cdot \underline{I}$$



$$\underline{Z} = \frac{\underline{U}}{\underline{I}} = R + j\omega L = Z \cdot e^{j\varphi_z}$$

$$Z = \sqrt{R^2 + (\omega L)^2}$$

$$\varphi = \arctan\frac{\omega L}{R}$$

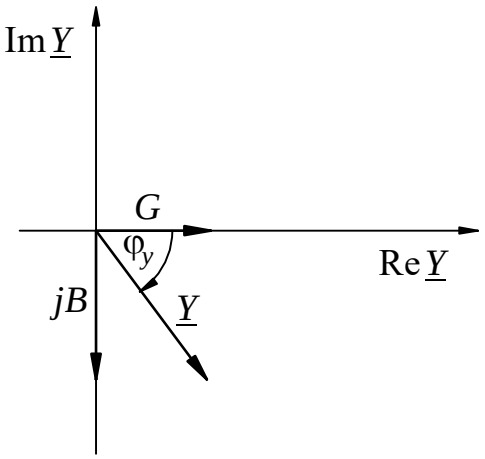


$$Y = \frac{1}{Z} \qquad \varphi_y = -\varphi$$

$$\underline{Y} = \frac{\underline{I}}{\underline{U}} = \frac{1}{R + j\omega L} \cdot \frac{R - j\omega L}{R - j\omega L}$$

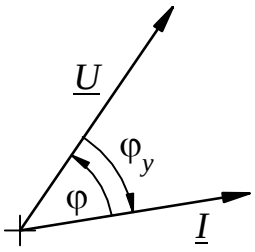
$$\underline{Y} = \frac{R}{R^2 + (\omega L)^2} + j\frac{-\omega L}{R^2 + (\omega L)^2}$$

$$G = \frac{R}{R^2 + (\omega L)^2} \qquad B = \frac{-\omega L}{R^2 + (\omega L)^2}$$



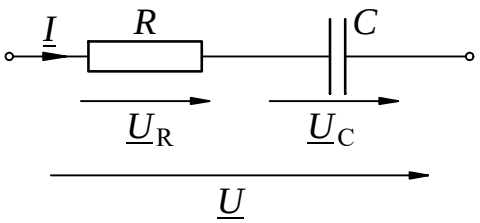
$$\underline{U} = \underline{Z} \cdot \underline{I} = Z \cdot I \cdot e^{j(\varphi_i + \varphi)}$$

$$\underline{I} = \underline{Y} \cdot \underline{U} = Y \cdot U \cdot e^{j(\varphi_u + \varphi_y)}$$



5.5.5 RC series circuit

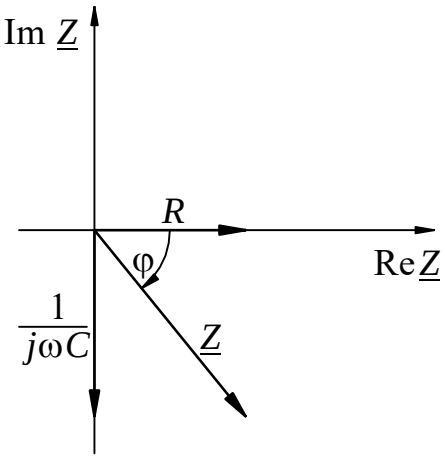
$$\underline{U} = \underline{U}_R + \underline{U}_C = R \cdot \underline{I} + \frac{1}{j\omega C} \cdot \underline{I}$$



$$\underline{Z} = \frac{\underline{U}}{\underline{I}} = R + \frac{1}{j\omega C} = R + j\left(\frac{-1}{\omega C}\right) = Z \cdot e^{j\varphi}$$

$$Z = \sqrt{R^2 + \left(\frac{1}{\omega C}\right)^2}$$

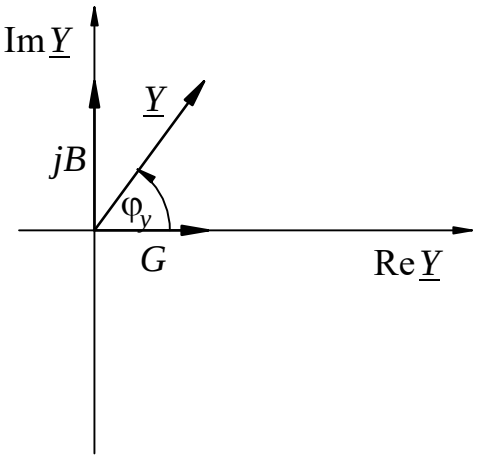
$$\varphi = \arctan\left(-\frac{1}{\omega C \cdot R}\right)$$



$$Y = \frac{1}{Z} \qquad \varphi_y = -\varphi$$

$$\underline{Y} = \frac{\underline{I}}{\underline{U}} = \frac{1}{R - j\frac{1}{\omega C}} \cdot \frac{R + j\frac{1}{\omega C}}{R + j\frac{1}{\omega C}}$$

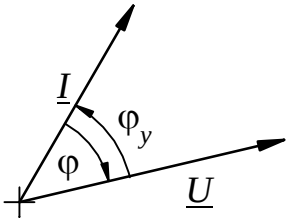
$$\underline{Y} = \frac{R}{R^2 + \frac{1}{\omega^2 C^2}} + j\frac{\frac{1}{\omega C}}{R^2 + \frac{1}{\omega^2 C^2}}$$



$$G = \frac{R}{R^2 + \frac{1}{\omega^2 C^2}} \qquad B = \frac{\frac{1}{\omega C}}{R^2 + \frac{1}{\omega^2 C^2}}$$

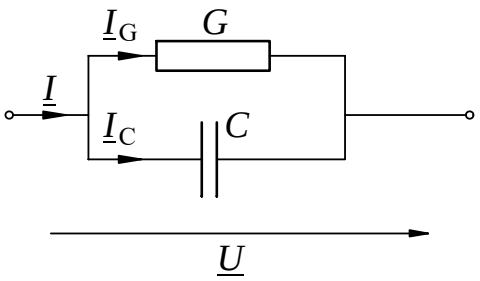
$$\underline{U} = \underline{Z} \cdot \underline{I} = Z \cdot I \cdot e^{j(\varphi_i + \varphi)}$$

$$\underline{I} = \underline{Y} \cdot \underline{U} = Y \cdot U \cdot e^{j(\varphi_u + \varphi_y)}$$



5.5.6 RC parallel circuit

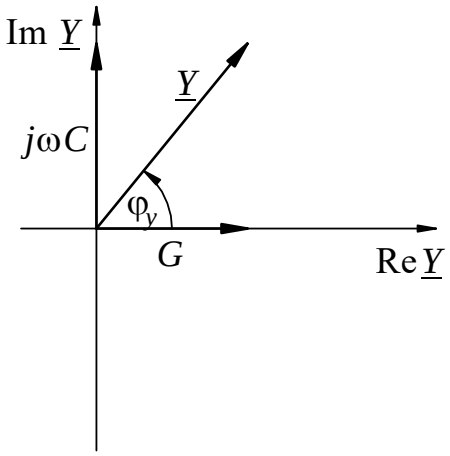
$$\underline{I} = \underline{I}_G + \underline{I}_C = G \cdot \underline{U} + j\omega C \cdot \underline{U}$$



$$\underline{Y} = \frac{\underline{I}}{\underline{U}} = G + j\omega C = Y \cdot e^{j\varphi_y}$$

$$Y = \sqrt{G^2 + (\omega C)^2}$$

$$\varphi_y = \arctan \frac{\omega C}{G}$$

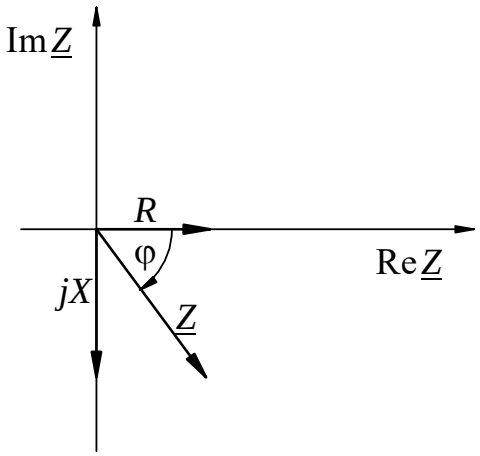


$$Z = \frac{1}{Y} \qquad \varphi = -\varphi_y$$

$$\underline{Z} = \frac{\underline{U}}{\underline{I}} = \frac{1}{G + j\omega C} \cdot \frac{G - j\omega C}{G - j\omega C}$$

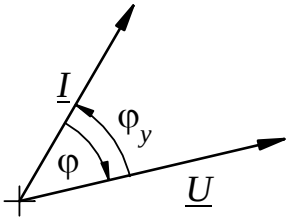
$$\underline{Z} = \frac{G}{G^2 + (\omega C)^2} + j\frac{-\omega C}{G^2 + (\omega C)^2}$$

$$R = \frac{G}{G^2 + (\omega C)^2} \qquad X = \frac{-\omega C}{G^2 + (\omega C)^2}$$



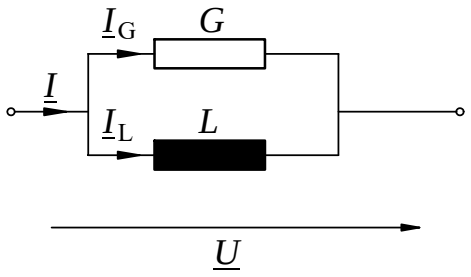
$$\underline{U} = \underline{Z} \cdot \underline{I} = Z \cdot I \cdot e^{j(\varphi_i + \varphi)}$$

$$\underline{I} = \underline{Y} \cdot \underline{U} = Y \cdot U \cdot e^{j(\varphi_u + \varphi_y)}$$



5.5.7 RL parallel circuit

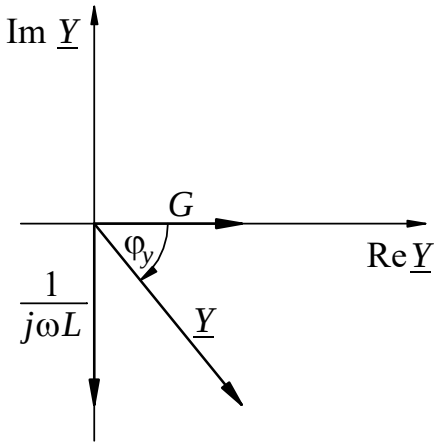
$$\underline{I} = \underline{I}_G + \underline{I}_L = G \cdot \underline{U} + \frac{1}{j\omega L} \cdot \underline{U}$$



$$\underline{Y} = \frac{\underline{I}}{\underline{U}} = G + \frac{1}{j\omega L} = G + j\left(\frac{-1}{\omega L}\right) = Y \cdot e^{j\varphi_y}$$

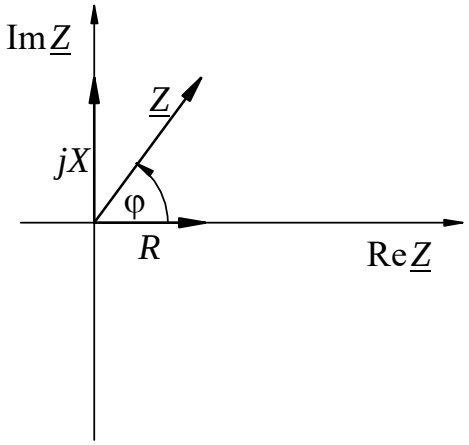
$$Y = \sqrt{G^2 + \left(\frac{1}{\omega L}\right)^2}$$

$$\varphi_y = \arctan\left(-\frac{1}{\omega L \cdot G}\right)$$



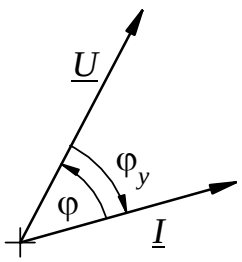
$$Z = \frac{1}{Y} \qquad \varphi = -\varphi_y$$

$$\underline{Z} = \frac{\underline{U}}{\underline{I}} = \frac{1}{G - j\frac{1}{\omega L}} \cdot \frac{G + j\frac{1}{\omega L}}{G + j\frac{1}{\omega L}}$$



$$\underline{Z} = \frac{G}{G^2 + \frac{1}{\omega^2 L^2}} + j\frac{\frac{1}{\omega L}}{G^2 + \frac{1}{\omega^2 L^2}}$$

$$R = \frac{G}{G^2 + \frac{1}{\omega^2 L^2}} \qquad X = \frac{\frac{1}{\omega L}}{G^2 + \frac{1}{\omega^2 L^2}}$$

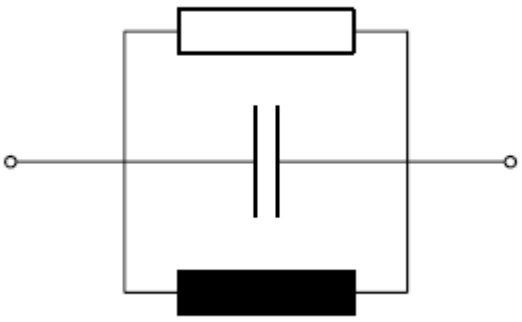


$$\underline{U} = \underline{Z} \cdot \underline{I} = Z \cdot I \cdot e^{j(\varphi_i + \varphi)}$$

$$\underline{I} = \underline{Y} \cdot \underline{U} = Y \cdot U \cdot e^{j(\varphi_u + \varphi_y)}$$

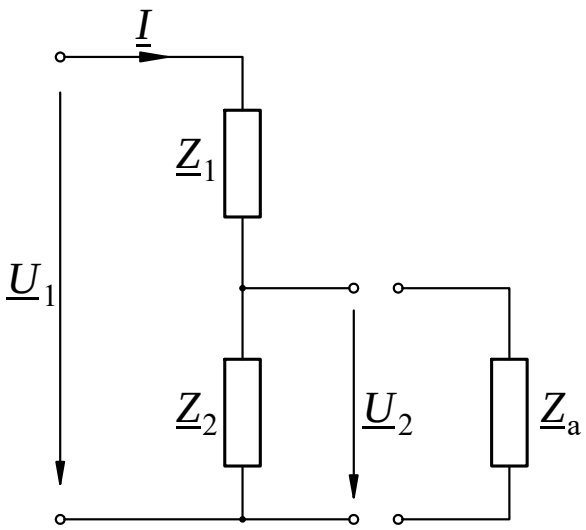
5.5.8 Dualities

series circuit	parallel circuit
open circuit	short circuit
ideal voltage source	ideal current source
complex voltage $\underline{U}$ RMS value U angle φ_u	complex current $\underline{I}$ RMS value I angle φ_i
complex resistance $\underline{Z}$ impedance Z phase angle φ resistance R reactance X	complex conductance $\underline{Y}$ admittance Y phase angle φ_y conductance G susceptance B
inductance L	capacitance C

Energy stored in a current carrying coil $W_m = \frac{1}{2} \cdot L \cdot I^2$	Energy stored in a charged capacitor $W_e = \frac{1}{2} \cdot C \cdot U^2$
RL series circuit $\underline{Z} = R + j\omega L$ $\underline{Y} = \frac{R}{R^2 + (\omega L)^2} + j\frac{-\omega L}{R^2 + (\omega L)^2}$	RC parallel circuit $\underline{Y} = G + j\omega C$ $\underline{Z} = \frac{G}{G^2 + (\omega C)^2} + j\frac{-\omega C}{G^2 + (\omega C)^2}$
series resonant circuit 	parallel resonant circuit 

5.6 Examples of AC circuits

5.6.1 Voltage divider



without load:

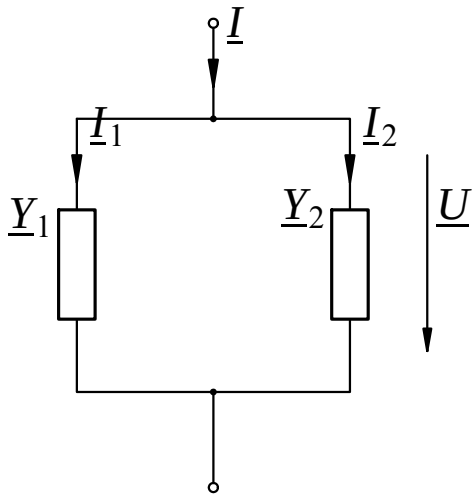
$$\frac{\underline{U}_2}{\underline{U}_1} = \frac{\underline{Z}_2 \cdot \underline{I}}{(\underline{Z}_1 + \underline{Z}_2) \cdot \underline{I}} = \frac{\underline{Z}_2}{\underline{Z}_1 + \underline{Z}_2}$$

like for DC current, but with complex quantities

with load:

$$\frac{\underline{U}_2}{\underline{U}_1} = \frac{\underline{Z}_2 \cdot \underline{Z}_a}{\underline{Z}_1 \cdot \underline{Z}_2 + \underline{Z}_1 \cdot \underline{Z}_a + \underline{Z}_2 \cdot \underline{Z}_a}$$

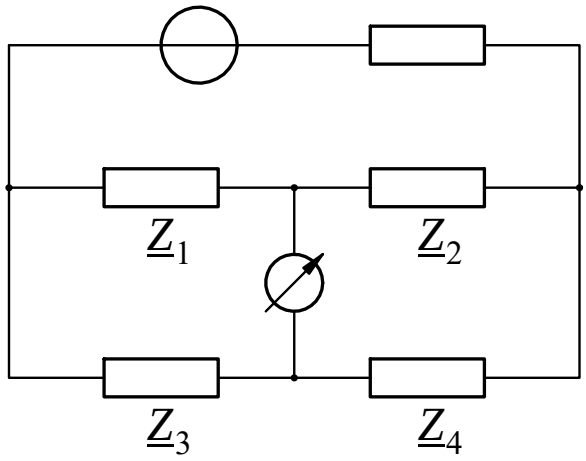
5.6.2 Current divider



$$\frac{\underline{I}_1}{\underline{I}} = \frac{\underline{Y}_1 \cdot \underline{U}}{(\underline{Y}_1 + \underline{Y}_2) \cdot \underline{U}} = \frac{\underline{Y}_1}{\underline{Y}_1 + \underline{Y}_2}$$

5.6.3 Bridge circuits

The next figure shows the basic layout for an AC bridge circuit.



The condition for balance is given in analogy to that for a DC bridge:

$$\frac{\underline{Z}_1}{\underline{Z}_2} = \frac{\underline{Z}_3}{\underline{Z}_4}$$

$$\underline{Z}_1 \cdot \underline{Z}_4 = \underline{Z}_2 \cdot \underline{Z}_3$$

If we use $\underline{Z} = R + jX$ we get:

$$(R_1 + jX_1) \cdot (R_4 + jX_4) = (R_2 + jX_2) \cdot (R_3 + jX_3)$$

$$R_1 \cdot R_4 + jR_1 \cdot X_4 + jR_4 \cdot X_1 - X_1 \cdot X_4 = R_2 \cdot R_3 + jR_2 \cdot X_3 + jR_3 \cdot X_2 - X_2 \cdot X_3$$

This equation is fulfilled, if the real parts and the imaginary parts on both sides are equal:

$$R_1 \cdot R_4 - X_1 \cdot X_4 = R_2 \cdot R_3 - X_2 \cdot X_3$$
$$R_1 \cdot X_4 + R_4 \cdot X_1 = R_2 \cdot X_3 + R_3 \cdot X_2$$

For an AC bridge obviously exist two conditions for balance, which means, that an AC bridge needs two adjustable elements to balance it.

The physical importance of these two conditions becomes more obvious, if we write the impedances in polar form:

$$\underline{Z}_1 \cdot \underline{Z}_4 = \underline{Z}_2 \cdot \underline{Z}_3 \qquad Z_1 \cdot Z_4 \cdot e^{j(\varphi_1 + \varphi_4)} = Z_2 \cdot Z_3 \cdot e^{j(\varphi_2 + \varphi_3)}$$

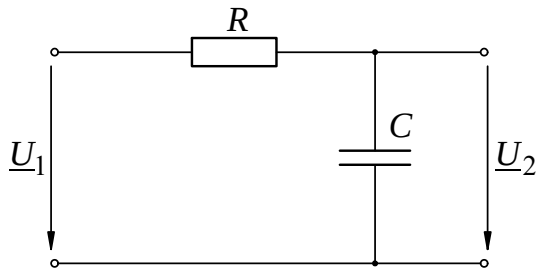
$$Z_1 \cdot Z_4 = Z_2 \cdot Z_3 \qquad \varphi_1 + \varphi_4 = \varphi_2 + \varphi_3$$

We see, that an AC bridge has to be adjusted (balanced) in terms of the magnitudes as well as in terms of the phase angles of its impedances.

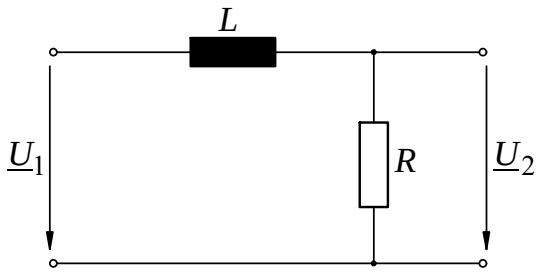
5.6.4 Low pass

A low pass circuit is a complex voltage divider, with a ratio of output to input voltage equal to one for low frequencies and equal to zero for high frequencies. The next figure shows two possible circuits.

a)



b)



From the condition for the voltage divider we get:

$$\frac{\underline{U}_2}{\underline{U}_1} = \frac{\frac{1}{j\omega C}}{R + \frac{1}{j\omega C}} = \frac{1}{1 + j\omega C \cdot R}$$

$$\frac{\underline{U}_2}{\underline{U}_1} = \frac{R}{R + j\omega L} = \frac{1}{1 + j\omega \cdot \frac{L}{R}}$$

The term τ is referred to as the time constant of the circuit.

$$\tau = R \cdot C$$

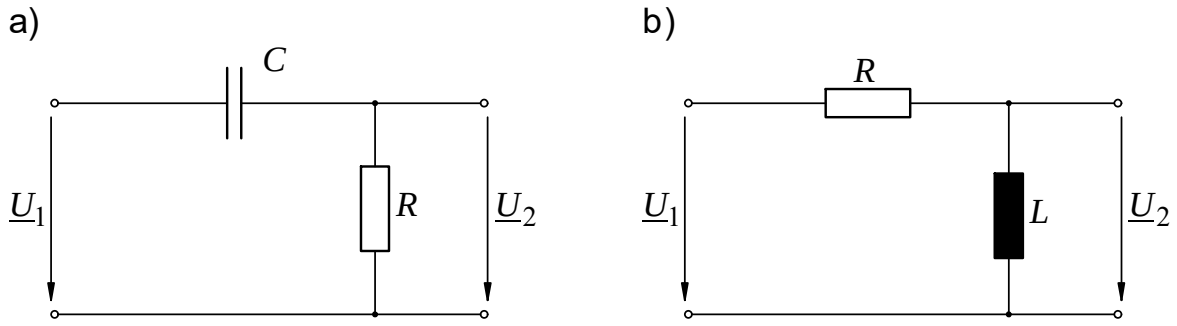
$$\tau = \frac{L}{R}$$

For ω = 0 s^{-1} the voltage ratio is one. For $\omega \to \infty$ the voltage ratio goes to zero.

As ideal inductances are not available, usually circuit a) is used to implement a low pass circuit.

5.6.5 High pass

A high pass circuit is a complex voltage divider, with a ratio of output to input voltage equal to one for high frequencies and equal to zero for low frequencies. The next figure shows two possible circuits.



From the condition for the voltage divider we get:

$$\frac{\underline{U}_2}{\underline{U}_1} = \frac{R}{R + \frac{1}{j\omega C}} = \frac{j\omega C \cdot R}{1 + j\omega C \cdot R}$$

$$\frac{\underline{U}_2}{\underline{U}_1} = \frac{j\omega L}{R + j\omega L} = \frac{j\omega \cdot \frac{L}{R}}{1 + j\omega \cdot \frac{L}{R}}$$

The term τ is again referred to as the time constant of the circuit.

$$\tau = R \cdot C$$

$$\tau = \frac{L}{R}$$

For ω = 0 s^{-1} the voltage ratio is zero. To see, how the circuit behaves for high frequencies, the equation describing the voltage ratio has to be rearranged:

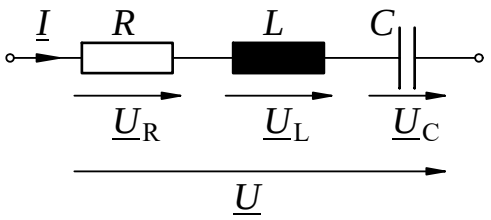
$$\frac{\underline{U}_2}{\underline{U}_1} = \frac{j\omega C \cdot R}{1 + j\omega C \cdot R} = \frac{1}{1 + \frac{1}{j\omega C \cdot R}}$$

Now it is obvious, that for $\omega \to \infty$ the voltage ratio goes to one.

For reasons mentioned above, usually circuit a) is used to implement a high pass behaviour..

5.6.6 Series resonant circuit

A series resonant circuit as shown here is a series connection of a resistance *R*, an inductance *L* and a capacitance *C*.

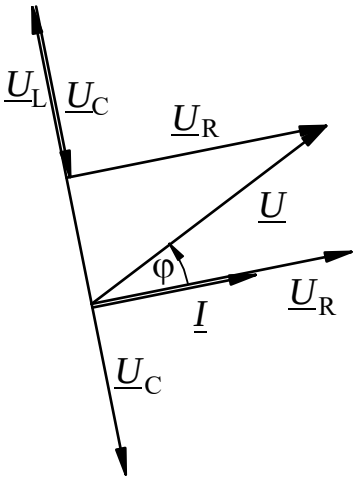


The total voltage $\underline{U}$ is the sum of the voltages across the three components:

$$\underline{U} = \underline{U}_R + \underline{U}_L + \underline{U}_C$$

$$\underline{U} = R \cdot \underline{I} + j\omega L \cdot \underline{I} + \frac{1}{j\omega C} \cdot \underline{I}$$

$$\underline{U} = R \cdot \underline{I} + j\left(\omega L \cdot \underline{I} - \frac{1}{\omega C} \cdot \underline{I} \right)$$



The impedance of the circuit is:

$$\underline{Z} = R + j\omega L + \frac{1}{j\omega C}$$

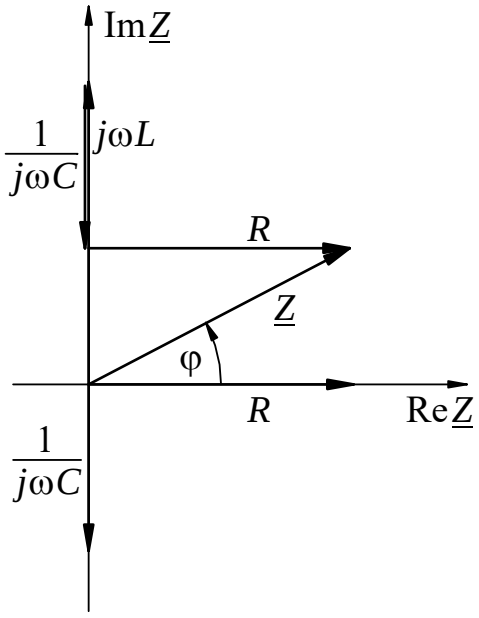
$$\underline{Z} = R + j\left(\omega L - \frac{1}{\omega C} \right)$$

The magnitude of the impedance is given by:

$$Z = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C} \right)^2}$$

The angle φ can be calculated as:

$$\varphi = \arctan\left(\frac{\omega L - \frac{1}{\omega C}}{R} \right)$$



Resonance

It is obvious, that for the angle φ it is possible $\varphi < 0$ as well as $\varphi > 0$. And of course, it is possible $\varphi = 0$. Then Resonance occurs, and the magnitude of the two reactances is the same:

$$\omega_0 L = \frac{1}{\omega_0 C}$$

The angular resonant frequency ω_0 as well as the resonant frequency f_0 are given by:

$$\boxed{\omega_0 = \frac{1}{\sqrt{L \cdot C}}} \qquad \boxed{f_0 = \frac{1}{2\pi \cdot \sqrt{L \cdot C}}}$$

The resonance is a voltage resonance, because:

$$\underline{U}_L + \underline{U}_C = 0$$
$$\underline{U}_L = -\underline{U}_C$$
$$U_L = U_C$$
$$\underline{U} = \underline{U}_R$$

The impedance of the circuit is $\underline{Z} = R$ and the characteristic impedance at the resonant frequency is:

$$\boxed{Z_0 = \omega_0 L = \frac{1}{\omega_0 C} = \sqrt{\frac{L}{C}}}$$

In case of resonance, the voltages across the three components can be calculated asfollows:

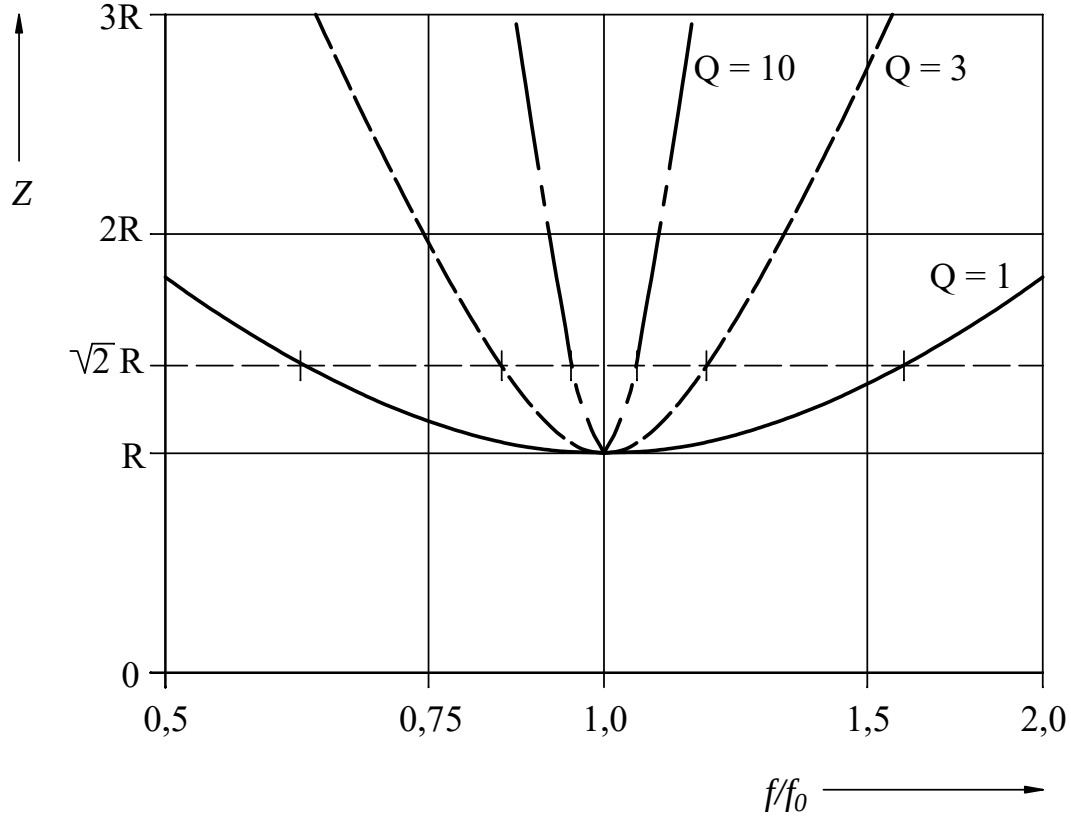
$$\underline{U}_R(\omega_0) = \underline{U} \qquad \underline{I}(\omega_0) = \frac{\underline{U}}{R}$$
$$\underline{U}_L = j\omega_0 L \cdot \underline{I}(\omega_0) = j\frac{\omega_0 L}{R} \cdot \underline{U} = j\frac{Z_0}{R} \cdot \underline{U}$$
$$\underline{U}_C = \frac{1}{j\omega_0 C} \cdot \underline{I}(\omega_0) = -j\frac{1}{\omega_0 C \cdot R} \cdot \underline{U} = -j\frac{Z_0}{R} \cdot \underline{U}$$

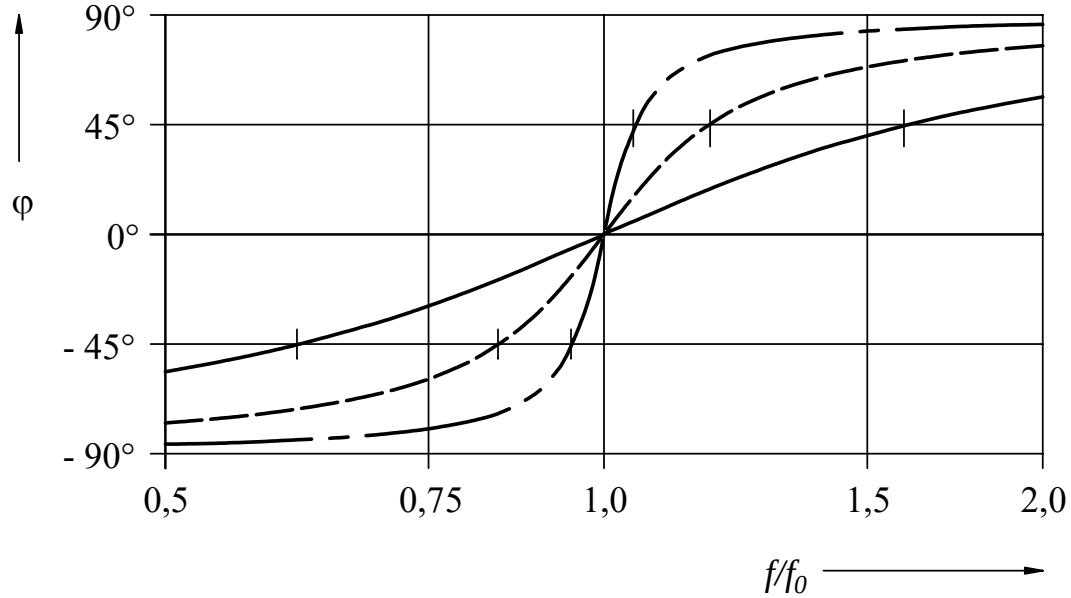
Quality factor

The sharpness of the resonance can be described by a factor of goodness or quality factor Q (Q-factor):

$$Q = \frac{U_L(\omega_0)}{U} = \frac{\omega_0 L}{R} \qquad Q = \frac{U_C(\omega_0)}{U} = \frac{1}{\omega_0 C \cdot R}$$
$$Q = \frac{Z_0}{R} = \frac{1}{R} \cdot \sqrt{\frac{L}{C}}$$

The reciprocal of the Q-factor is called the loss factor: $d = \frac{1}{Q}$





capacitive | inductive

Values for the Q-factor can be in the order of 10 to 20. Therefore the voltages U_L and U_C at frequencies close to the resonant frequency can be much higher than the voltage applied to the resonant circuit and the components *L* and *C* have to be designed for voltages that high.

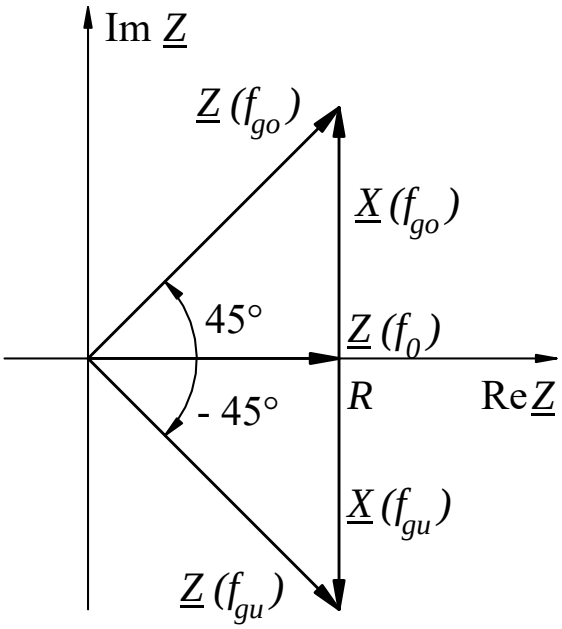
The figure on the previous page shows the magnitude and the phase angle of a series resonant circuit as a function of the ratio f/f_0 for different values of the *Q*-factor. It can be seen, that for large values of the *Q*-factor the magnitude of the impedance increases strongly for only small changes in frequency. This is why the *Q*-factor is also called sharpness of resonance.

Cut-off frequencies

The series resonant circuit at resonance has a rather low impedance

$$Z(f_0) = R$$

that becomes larger for lower and higher frequencies. The frequencies, for which the impedance is larger than R by a factor of $\sqrt{2}$, are called cut-off frequencies f_g. These cut-off frequencies are marked in the phasor diagram of the next figure.



The magnitude of Z at the cut-off frequencies is:

$$Z(f_g) = \sqrt{2} \cdot R$$
$$\Rightarrow \quad X = \pm R$$

The phase angle for the cut-off frequencies is ± 45°. That is why the cut-off frequencies are also referred to as 45°- frequencies.

The band width Δf of a resonant circuit is defined as the difference between theupper and lower cut-off frequency:

$$\Delta f = f_{go} - f_{gu}$$

Further calculations:

$$\frac{X}{R} = +/-1$$
$$Q \cdot \left(\frac{f_g}{f_0} - \frac{f_0}{f_g} \right) = +/-1$$

$$\frac{f_g}{f_0} - \frac{f_0}{f_g} \mp \frac{1}{Q} = 0 \qquad \left| \times f_0 f_g \right.$$

$$f_g^2 \mp \frac{f_0}{Q} \cdot f_g - f_0^2 = 0$$

$$f_g = \pm \frac{f_0}{2Q} \pm \sqrt{\frac{f_0^2}{4 \cdot Q^2} + f_0^2}$$

higher cut-off frequency $\quad f_{go} = f_0 \cdot \left(\sqrt{1 + \frac{1}{4Q^2}} + \frac{1}{2Q} \right)$

lower cut-off frequency $\quad f_{gu} = f_0 \cdot \left(\sqrt{1 + \frac{1}{4Q^2}} - \frac{1}{2Q} \right)$

$$\Delta f = f_{go} - f_{gu} \qquad f_{go} \cdot f_{gu} = f_0 \cdot \left(W + \frac{1}{2Q} \right) \cdot f_0 \cdot \left(W - \frac{1}{2Q} \right)$$

$$\Delta f = f_0 \cdot \left[\left(\sqrt{1 + \frac{1}{4Q^2}} + \frac{1}{2Q} \right) - \left(\sqrt{1 + \frac{1}{4Q^2}} - \frac{1}{2Q} \right) \right] \qquad \text{mit} \quad W = \sqrt{1 + \frac{1}{4Q^2}}$$

$$\Delta f = f_0 \cdot \frac{1}{Q} \qquad f_{go} \cdot f_{gu} = f_0^2 \cdot \left(W^2 - \frac{1}{4Q^2} \right)$$

$$f_{go} \cdot f_{gu} = f_0^2 \cdot \left(1 + \frac{1}{4Q^2} - \frac{1}{4Q^2} \right)$$

$$\boxed{Q = \frac{f_0}{\Delta f}} \qquad \boxed{f_0 = \sqrt{f_{go} \cdot f_{gu}}}$$

With this bandwidth also the *Q*-factor can be expressed, showing again, why *Q* is also called sharpness of resonance.

The resonant frequency is the geometric mean of the two cut-off frequencies

Frequencies for the voltage maxima

The next figure shows once more a series resonant circuit excited by a constant voltage *U*. In addition diagrams showing current, voltage and impedance for the three components as a function of the frequency are given.

For the impedance and ist magnitude we get:

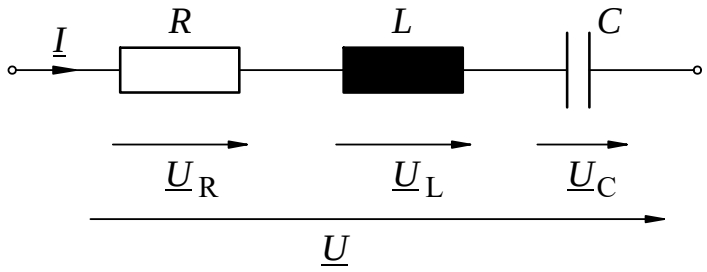
$$\underline{Z} = R + j\left(\omega L - \frac{1}{\omega C} \right) = R \cdot \left[1 + j\left(\frac{\omega_0 L}{R} \cdot \frac{\omega}{\omega_0} - \frac{1}{\omega_0 C \cdot R} \cdot \frac{\omega_0}{\omega} \right) \right] = R \cdot \left[1 + jQ \cdot \left(\frac{f}{f_0} - \frac{f_0}{f} \right) \right]$$

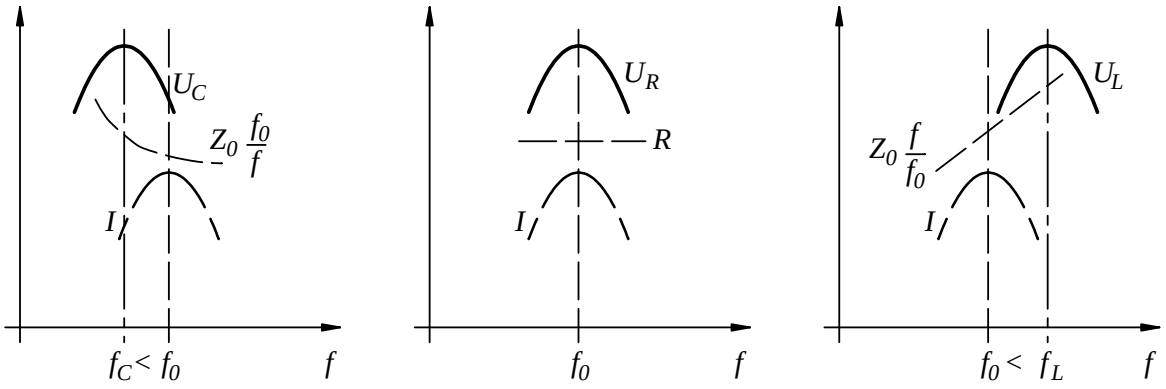
$$Z = R \cdot \sqrt{1 + Q^2 \cdot \left(\frac{f}{f_0} - \frac{f_0}{f} \right)^2}$$

The magnitude of the current is given by:

$$I = \frac{U}{Z} = \frac{U}{R \cdot \sqrt{1 + Q^2 \cdot \left(\frac{f}{f_0} - \frac{f_0}{f} \right)^2}}$$

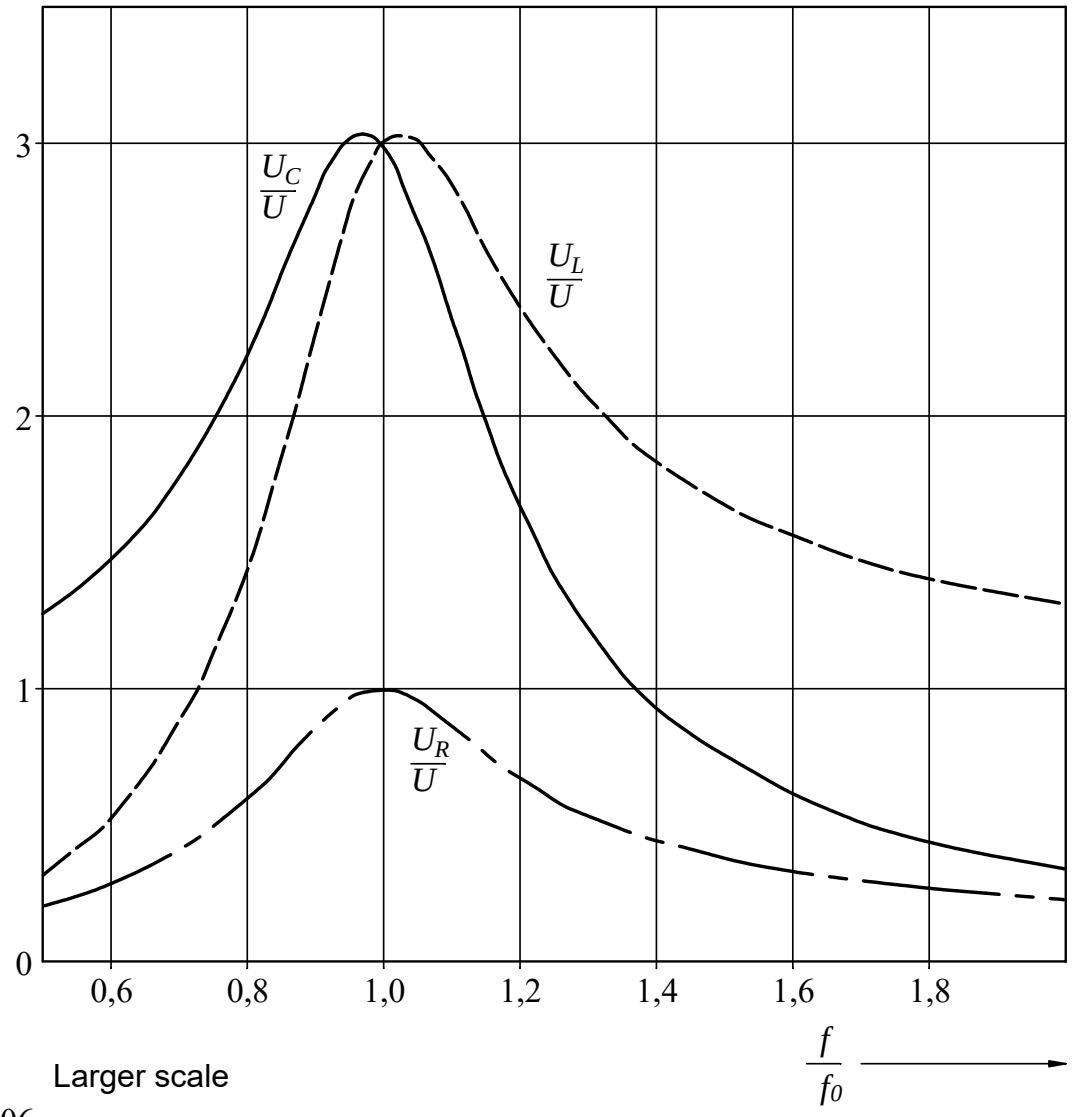
It has its maximum at $f = f_0$ because $Z(f)$ has its minimum there.



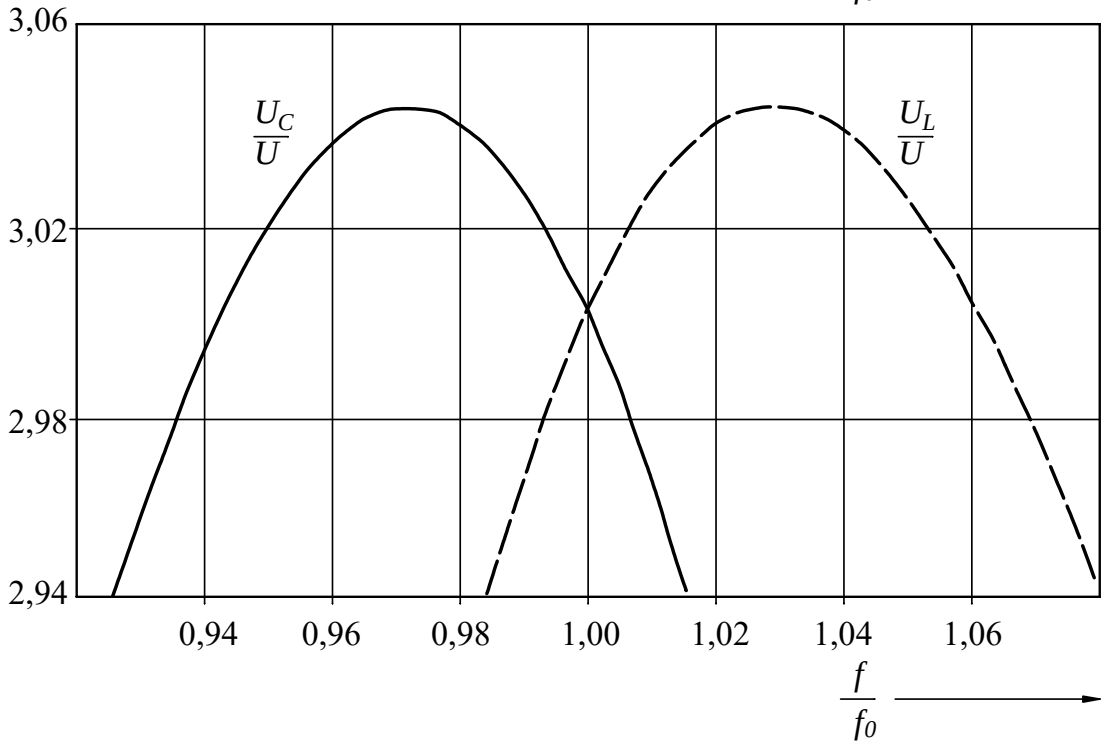


The maximum values of the voltages $U_L(f)$ and $U_C(f)$ are somewhat shifted against the current maximum due to the frequency dependence of Z. The maximum of the voltage $U_R(f)$, however, occurs at the resonance frequency f_0.

The figure at the next page shows the relations once more for a series resonant circuit with the Q-factor 3, which is applied to a constant voltage.

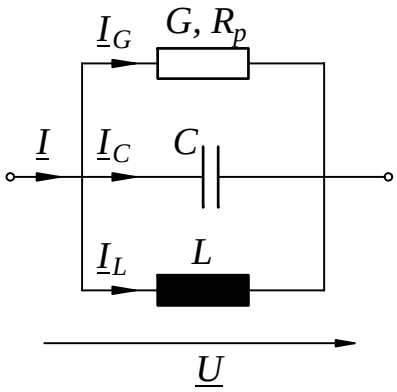


Larger scale



5.6.7 Parallel resonant circuit

A parallel resonant circuit as shown here is a parallel connection of a conductance *G*, a capacitance *C* and an inductance *L*

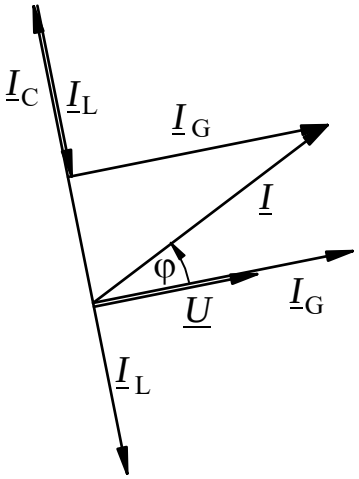


The total current *I* is the sum of the currents through the three components:

$$\underline{I} = \underline{I}_G + \underline{I}_C + \underline{I}_L$$

$$\underline{I} = G \cdot \underline{U} + j\omega C \cdot \underline{U} + \frac{1}{j\omega L} \cdot \underline{U}$$

$$\underline{I} = \left(G + j\omega C + \frac{1}{j\omega L} \right) \cdot \underline{U}$$



The admittance of the circuit is:

$$\underline{Y} = G + j\omega C + \frac{1}{j\omega L}$$

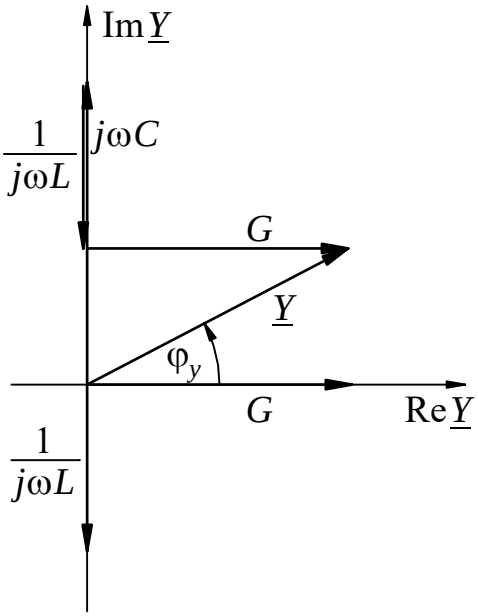
$$\underline{Y} = G + j\left(\omega C - \frac{1}{\omega L} \right)$$

The magnitude of the admittance is given by:

$$Y = \sqrt{G^2 + \left(\omega C - \frac{1}{\omega L} \right)^2}$$

The angle φ can be calculated as:

$$\varphi_y = \arctan\left(\frac{\omega C - \frac{1}{\omega L}}{G} \right)$$



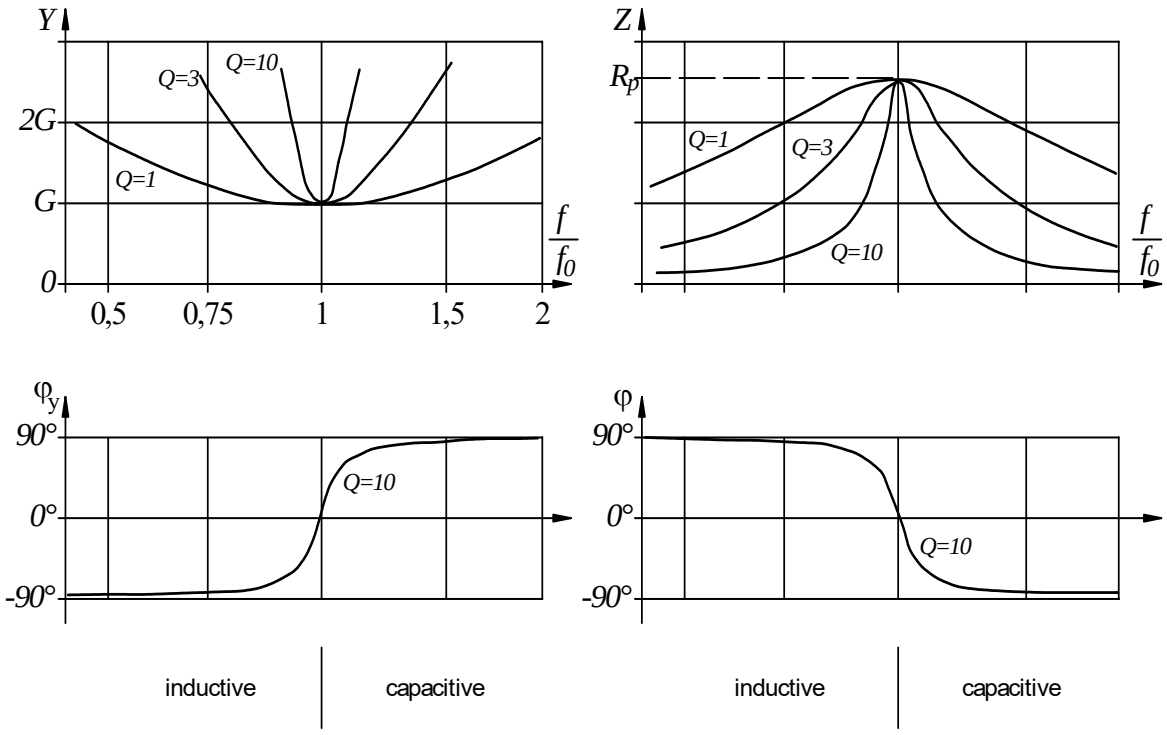
The following table shows the dualities between the series and the parallel resonant circuit:

	series resonant circuit	parallel resonant circuit
angular resonant frequency	$\omega_0 = \frac{1}{\sqrt{LC}}$	$\omega_0 = \frac{1}{\sqrt{CL}}$
characteristic impedance	$Z_0 = \omega_0 L = \sqrt{\frac{L}{C}}$	$Z_0 = \omega_0 L = \sqrt{\frac{L}{C}}$
characteristic admittance	$Y_0 = \omega_0 C = \sqrt{\frac{C}{L}}$	$Y_0 = \omega_0 C = \sqrt{\frac{C}{L}}$
Q-factor, sharpness of resonance	$Q = \frac{\omega_0 L}{R} = \frac{Z_0}{R}$	$Q = \frac{\omega_0 C}{G} = \frac{Y_0}{G} = \frac{R_p}{Z_0}$

Because of the dualities shown above, magnitude and phase angle of the admittance *Y* as a function of the frequency ratio f/f_0 are curves similar to the curves for *Z* for the series resonant circuit. Magnitude and phase angle curves for the impedance *Z* can be derived from that using:

$$\underline{Z} = \frac{1}{\underline{Y}} \qquad Z = \frac{1}{Y} \qquad \varphi = -\varphi_y$$

Magnitude and phase angle of the admittance and the impedance of a parallel resonant circuit around its resonant frequency are shown below.



5.7 Power in AC circuits

5.7.1 Instantaneous power

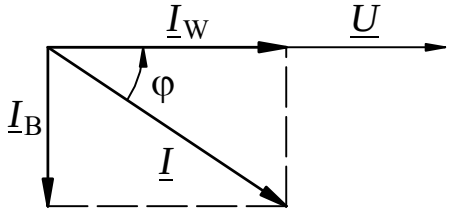
The instantaneous power is the product of the instantaneous voltage and the instantaneous current:

$$p(t) = u(t) \cdot i(t)$$

Depending on the signs of voltage and current it can have a negative sign, which means that the flow of energy changes its direction.

We consider two sinusoidal alternating quantities *u* and *i*, that have a phase shift of $\varphi = \varphi_u - \varphi_i$. We choose arbitrarily $\varphi_u = 0$:

$$u(t) = \hat{u} \cdot \sin(\omega t)$$
$$i(t) = \hat{i} \cdot \sin(\omega t - \varphi)$$
$$i(t) = \hat{i} \cdot \cos\varphi \cdot \sin(\omega t) + i \cdot \sin\varphi \cdot \left[-\cos(\omega t)\right]$$
$$i(t) = \hat{i}_W \cdot \sin(\omega t) + i_B \cdot \left[-\cos(\omega t)\right]$$



Complex notation:

$$\underline{I} = \underline{I}_W + \underline{I}_B$$

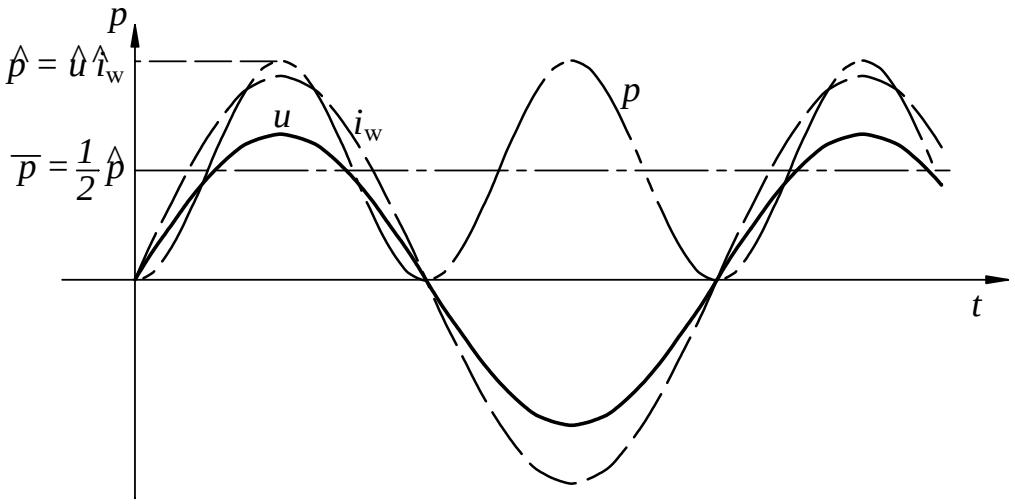
$I_W = I \cdot \cos\varphi$ active current

$I_B = I \cdot \sin\varphi$ reactive current

5.7.2 Active power

The active power is the arithmetic mean of the instantaneous power. The calculation of this mean value takes into account only the active current (see chapter 5.7.4 for the reason).

$$P = \overline{p(t)} = \frac{1}{T} \cdot \int_{t_0}^{t_0+T} u(t) \cdot i(t) dt$$



$$P = \overline{p} = \frac{\hat{u} \cdot \hat{i}_W}{2} = \frac{\hat{u}}{\sqrt{2}} \cdot \frac{\hat{i}_W}{\sqrt{2}} = U \cdot I_W$$

because $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$ (see 5.1.4)

Active power: $\boxed{P = U \cdot I \cdot \cos\varphi}$ $\cos\varphi$: power factor

Active power is power in a physical sense = the mean work performed per unit of time

Unit of the active power: [P] = 1 W (Watt)

At a resistance only active power is dissipated!

5.7.3 Apparent power

The apparent power (also: complex power) is the product of the RMS - values of current and voltage.

Apparent power: $\boxed{S = U \cdot I}$

General definition of the power factor: $\lambda = \frac{P}{S}$

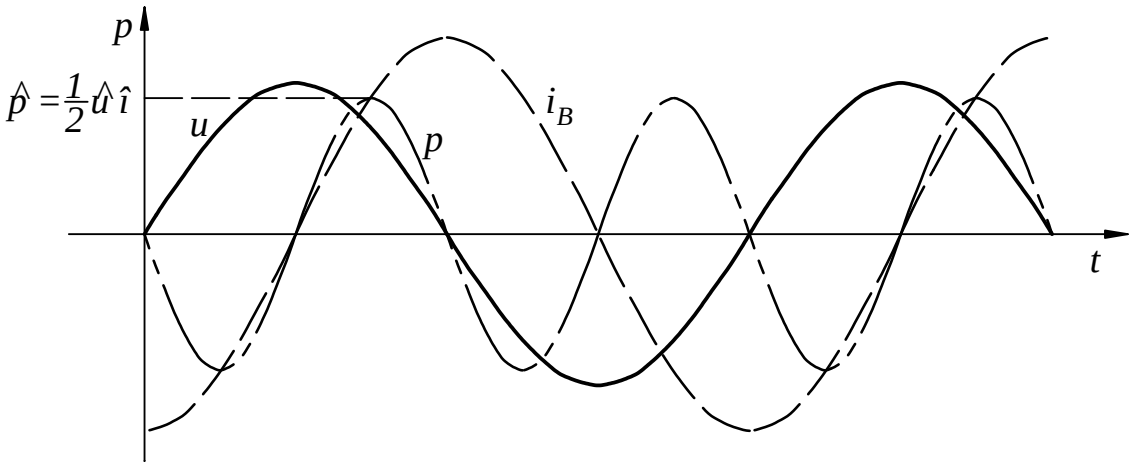
Power factor for sinusoidal voltages and currents: $\lambda = \frac{P}{S} = \cos\varphi$

The apparent power is merely a mathematical operand without a direct physical meaning.

Unit of the apparent power: [S] = 1 VA (Volt-Ampere)

5.7.4 Reactive power

The interaction between voltage and reactive current also results in a flow of energy → oscillating power



Energy is oscillating (e.g. between source and load) but is not transformed into another kind of energy like mechanical work or heat (not dissipated).

The arithmetic mean of this instantaneous power is zero. In calculating the active power, there is obviously no contribution of the reactive current i_B.

Definition: $Q = U \cdot I_B$

Reactive power: $\boxed{Q = U \cdot I \cdot \sin\varphi}$

Reactive power is not power in a physical sense.

Unit of the reactive power:[Q] = 1 var (Volt-Ampere reactive)

At ideal reactances (*L, C*) there is only reactive power!

5.7.5 Complex power calculation

So far we have seen the following relations between *S, P* and *Q*:

$\boxed{P = U \cdot I \cdot \cos\varphi}$ $\qquad$ $\boxed{S = \sqrt{P^2 + Q^2}}$

$$S^2 = P^2 + Q^2$$

$$\tan\varphi = \frac{Q}{P}$$

$\boxed{Q = U \cdot I \cdot \sin\varphi}$ $\qquad$ $\boxed{\varphi = \arctan\left(\frac{Q}{P}\right)}$

These relations are of the same kind as the relations between *Z, R* and *X* .So the definition of a complex phasor to represent power is reasonable:

$$\underline{S} = P + jQ$$

To determine *P* and *Q*, we need the magnitudes *I* and *U* as well as their phase angle difference $\varphi = \varphi_u - \varphi_i$.

As a first approach it looks reasonable, to define a complex alternating power as:

$$\underline{S}_{\sim} = \underline{U} \cdot \underline{I} = U \cdot e^{j\varphi_u} \cdot I \cdot e^{j\varphi_i} = U \cdot I \cdot e^{j(\varphi_u + \varphi_i)}$$

But the result of this approach contains the sum of the phase angles and not the difference. For that reason we define the complex power as the product of the complex voltage and the conjugate of the complex current:

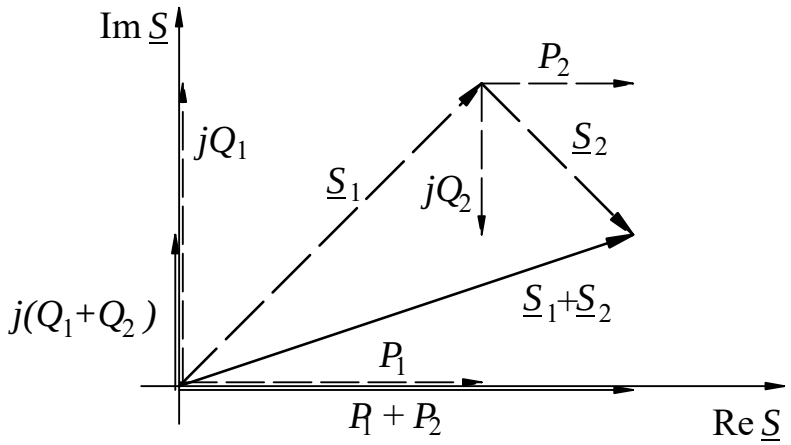
$$\boxed{\underline{S} = \underline{U} \cdot \underline{I}^{*} = S \cdot e^{j\varphi}}$$

$$\underline{S} = U \cdot e^{j\varphi_u} \cdot I \cdot e^{-j\varphi_i} = U \cdot I \cdot e^{j(\varphi_u - \varphi_i)}$$

The components of this complex phasor are:

$$P = \mathrm{Re}\,\underline{S} = S \cdot \cos\varphi = U \cdot I \cdot \cos\varphi$$
$$Q = \mathrm{Im}\,\underline{S} = S \cdot \sin\varphi = U \cdot I \cdot \sin\varphi$$

Complex powers are added geometrically:



$$\underline{S}_1 + \underline{S}_2 = (P_1 + jQ_1) + (P_2 + jQ_2) = (P_1 + P_2) + j(Q_1 + Q_2)$$

The apparent or complex power at an impedance can be calculated as follows:

$$\underline{S} = \underline{U} \cdot \underline{I}^{*} = U \cdot e^{j\varphi_u} \cdot I \cdot e^{-j\varphi_i} = U \cdot I \cdot e^{j\varphi} = (Z \cdot I) \cdot I \cdot e^{j\varphi} = I^2 \cdot Z \cdot e^{j\varphi}$$

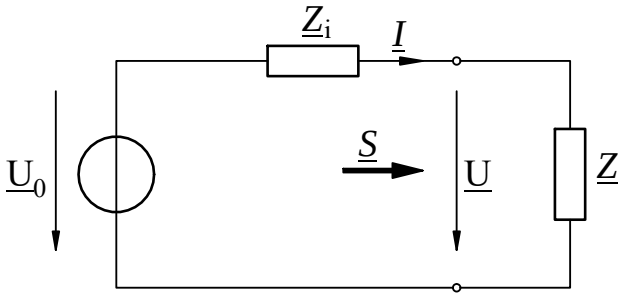
$$\underline{S} = \underline{Z} \cdot I^2$$

If the voltage is used instead of the current, we get:

$$\underline{S} = \underline{U} \cdot \underline{I}^* = U \cdot I \cdot e^{j\varphi} = U \cdot \left(\frac{U}{Z}\right) \cdot e^{j\varphi} = \frac{U^2}{Z \cdot e^{-j\varphi}}$$

$$\underline{S} = \frac{U^2}{\underline{Z}^*} = \underline{Y}^* \cdot U^2$$

Complex power at an impedance:



$$\underline{S} = (R + jX) \cdot I^2$$

$$P = \mathrm{Re}\,\underline{S} = R \cdot I^2$$

$$Q = \mathrm{Im}\,\underline{S} = X \cdot I^2$$

for an ohmic-inductive load:

$$\underline{Z} = R + j\omega L$$

$$\underline{S} = (R + j\omega L) \cdot I^2$$

$$P = R \cdot I^2$$

$$Q = \omega L \cdot I^2$$ positive!

for an ohmic-capacitive load:

$$\underline{Z} = R + \frac{1}{j\omega C}$$

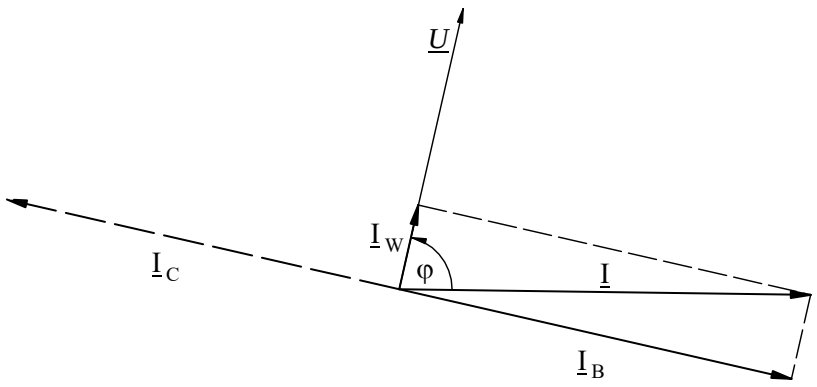
$$\underline{S} = \left(R + j\frac{-1}{\omega C}\right) \cdot I^2$$

$$P = R \cdot I^2$$

$$Q = \frac{-1}{\omega C} \cdot I^2$$ negative!

5.7.6 Reactive power compensation

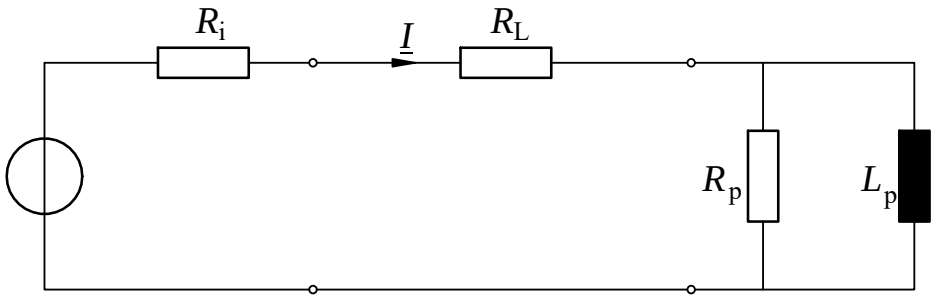
The following figure shows the current-voltage phasor diagram for a load with a large inductance.



Only a small fraction $I_W = I \cdot \cos\varphi$ of the total current *I* contributes to the active power:

$$P = U \cdot I \cdot \cos\varphi$$

Only this fraction can perform work and has to be payed for. But the total current *I* has to be transported from the power station to the load. This leads to energy losses in the power lines. The next figure shows this situation in an equivalent circuit, containing also the line resistance R_L.



Line losses: $P_V = R_L \cdot I^2$

These line losses can be reduced if the reactive part of the current, $I_B = I \cdot \sin\varphi$, that "produces" only reactive power $Q = U \cdot I \cdot \sin\varphi$ at the load, can be reduced.

This can be done by adding a capacitor parallel to the load, that takes a current $\underline{I}_C = -\underline{I}_B$.

The current $\underline{I}_C$ must lead the voltage by 90° → capacitance

The current $\underline{I}_C$ must be added to the currents $\underline{I}$ → parallel circuit

By the parallel capacitance the reactive power of the whole circuit is to be compensated to zero.

$$Q + Q_C = 0$$

$$Q_C = -Q$$

$$\mathrm{Im}\left\{\underline{Y}_C^* \cdot U^2\right\} = -Q$$

$$-\omega C \cdot U^2 = -Q$$

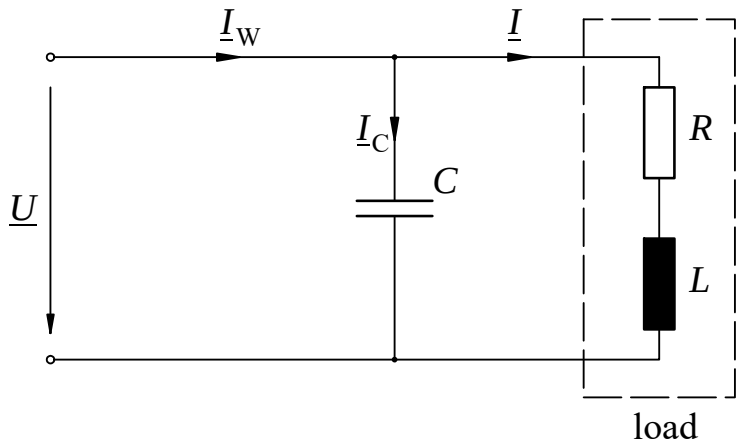
without using complex power:

$$Q_C = X_C \cdot {I_C}^2 = -\frac{1}{\omega C} \cdot {I_C}^2$$

$$I_C = \omega C \cdot U$$

$$Q_C = -\omega C \cdot U^2$$

$$\boxed{C = \frac{Q}{\omega \cdot U^2}}$$


$\underline{I}_W$
$\underline{I}$
$\underline{I}_C$
C
R
$\underline{U}$
L
load

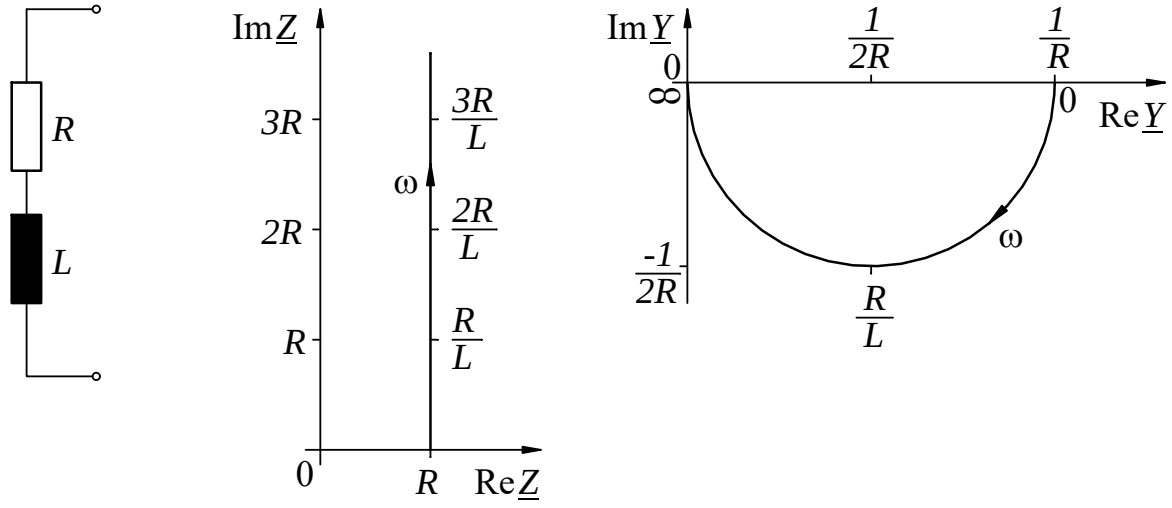
6. Loci

A diagram that shows a complex quantity as a function of a variable in the Gaussian plane is called a locus. In electrical engineering this variable is usually the frequency *f* or the angular frequency ω.

6.1 Loci of RL and RC circuits

6.1.1 RL series circuit

Loci of impedance and admittance of an RL - series circuit:



The impedance of the circuit is:

$$\underline{Z} = R + j\omega L$$

$$\underline{Z} = Z \cdot e^{j\varphi}$$

The admittance is the reciprocal of the impedance:

$$\underline{Y} = \frac{1}{\underline{Z}} = \frac{1}{Z} \cdot e^{-j\varphi}$$

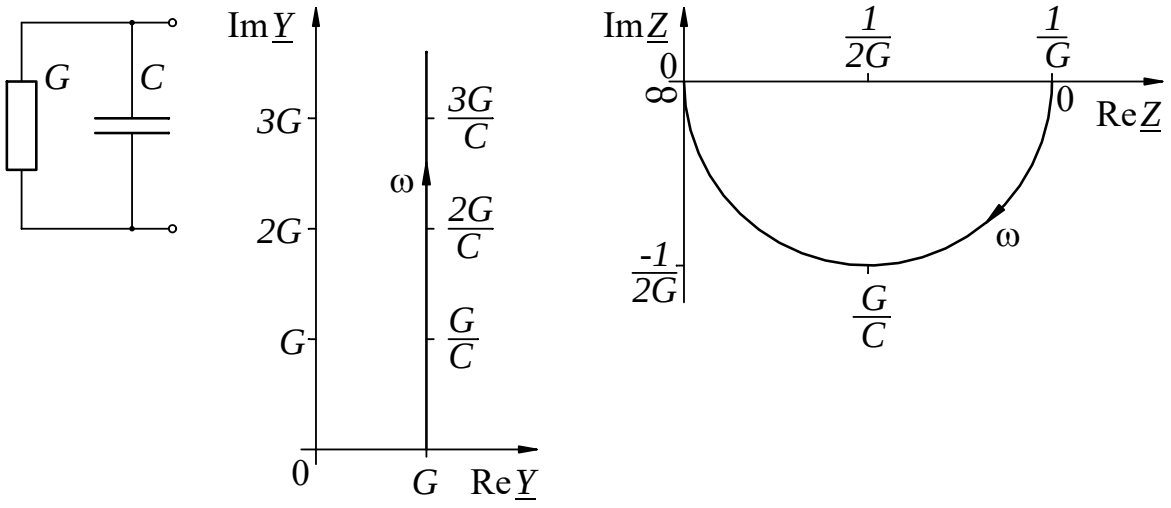
For a phasor this inversion can be done by taking the reciprocal of the magnitude and changing the sign of the angle φ. This results in a semicircle as locus of the admittance.

The following table gives values for *Z* and φ, *Y* and φ_Y for some characteristic values of ω.

$\omega = 0$	$Z = R$	$\varphi = 0°$	$Y = \frac{1}{R}$	$\varphi_y = 0°$
$\omega = \frac{R}{L}$	$Z = \sqrt{2} \cdot R$	$\varphi = 45°$	$Y = \frac{1}{\sqrt{2} \cdot R}$	$\varphi_y = -45°$
$\omega \to \infty$	$Z \to \infty$	$\varphi \to 90°$	$Y = 0$	$\varphi_y \to -90°$

6.1.2 GC parallel circuit

Loci of admittance and impedance of an RC - parallel circuit. For an easier representation instead of the ohmic resistance *R* its conductance *G* is used:



The admittance of the circuit is:

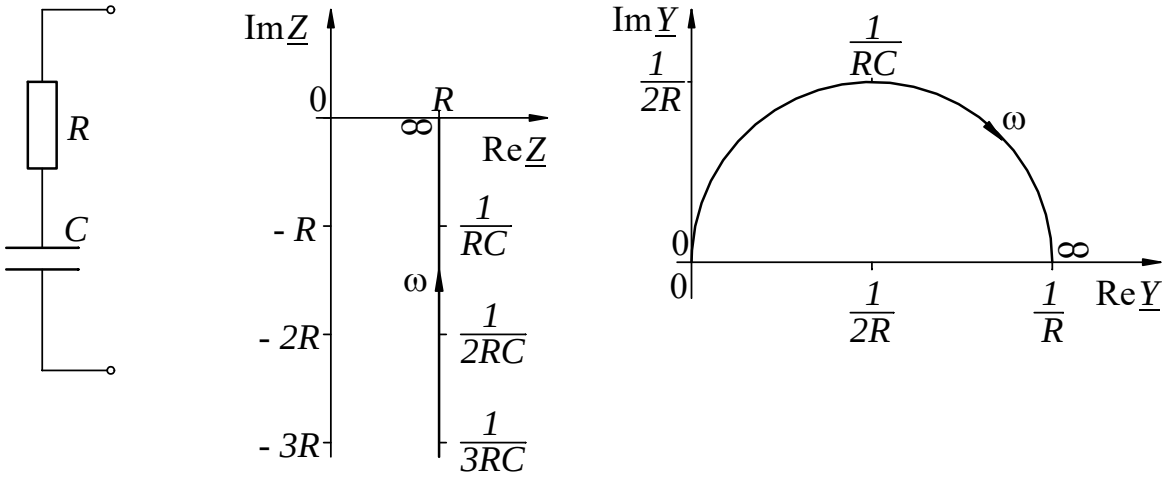
$$\underline{Y} = G + j\omega C$$

The following table gives the dualities for the RL - series and the GC - parallel circuit:

series circuit	parallel circuit
R	*G*
L	*C*
$\underline{Z}$	$\underline{Y}$

6.1.3 RC series circuit

Loci of impedance and admittance of an RC - series circuit:



The impedance of the circuit is:

$$\underline{Z} = R + \frac{1}{j\omega C} = R - j\frac{1}{\omega C}$$

$$\underline{Z} = Z \cdot e^{j\varphi}$$

The admittance is the reciprocal of the impedance:

$$\underline{Y} = \frac{1}{\underline{Z}} = \frac{1}{Z} \cdot e^{-j\varphi}$$

For a phasor this inversion can be done by taking the reciprocal of the magnitude and changing the sign of the angle φ. This results in a semicircle as locus of the admittance.

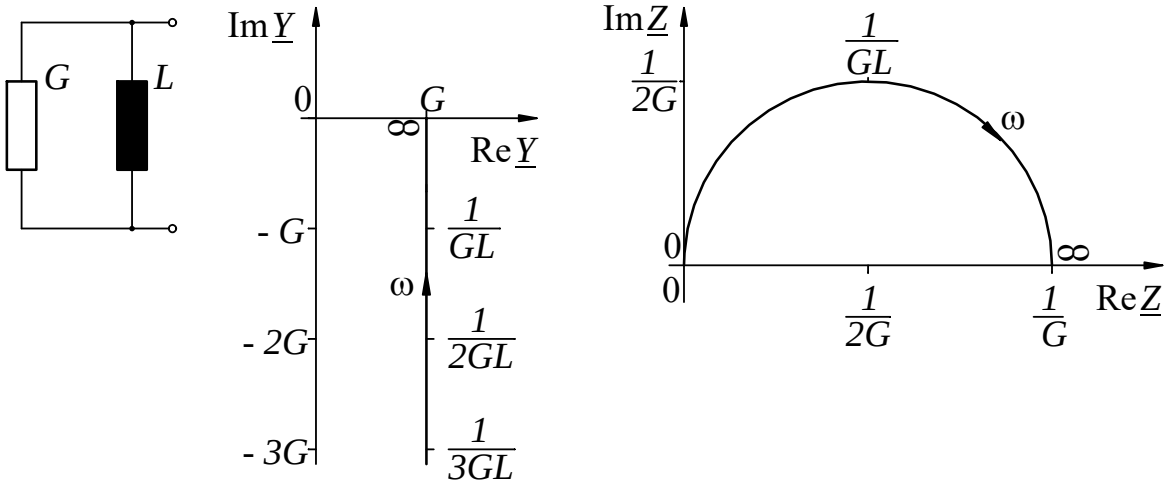
The following table gives values for Z and φ, Y and φ_y for some characteristic values of ω.

.

$\omega = 0$	$Z \to \infty$	$\varphi \to -90°$	$Y = 0$	$\varphi_y \to 90°$
$\omega = \frac{1}{RC}$	$Z = \sqrt{2} \cdot R$	$\varphi = -45°$	$Y = \frac{1}{\sqrt{2} \cdot R}$	$\varphi_y = 45°$
$\omega \to \infty$	$Z = R$	$\varphi = 0°$	$Y = \frac{1}{R}$	$\varphi_y = 0°$

6.1.4 GL parallel circuit

Loci of admittance and impedance of an GL - parallel circuit:



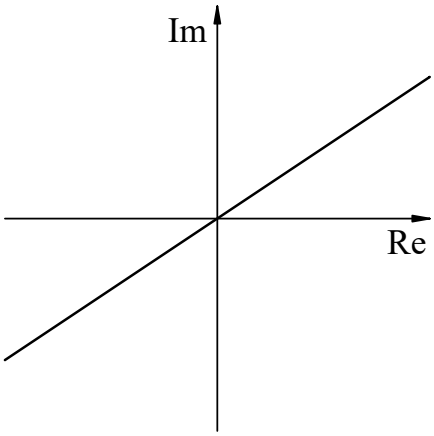
The admittance of the circuit is:

$$\underline{Y} = G + \frac{1}{j\omega L} = G - j\frac{1}{\omega L}$$

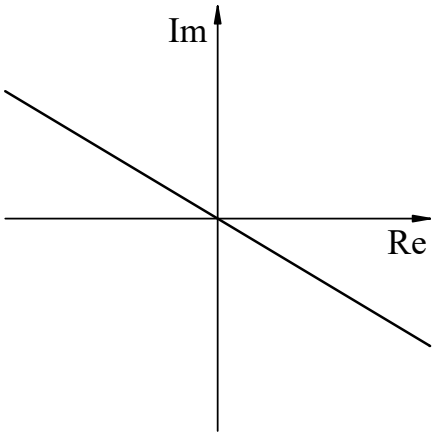
The following table gives the dualities for the RL - series and the GC - parallel circuit:

series circuit	parallel circuit
R	G
C	L
$\underline{Z}$	$\underline{Y}$

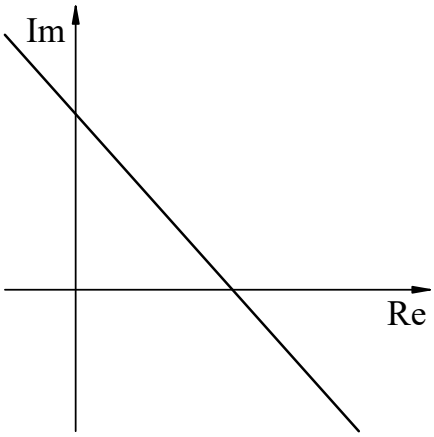
6.2 Inversion of loci



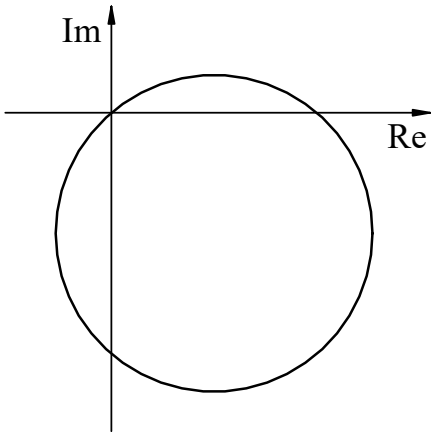
line through the origin



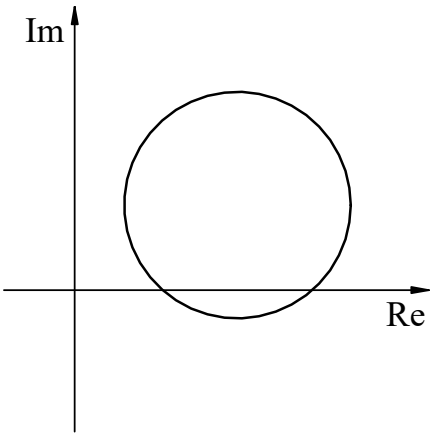
line through the origin



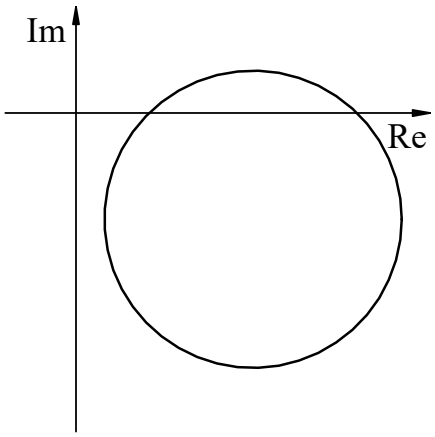
line not through the origin



circle through the origin



circle not through the origin



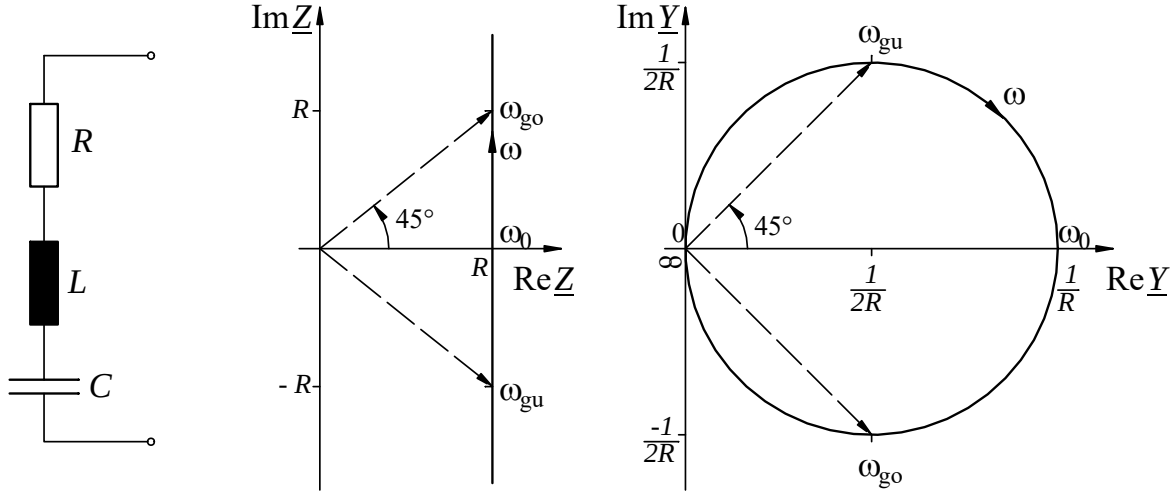
circle not through the origin

Some common rules for the inversion of loci.

6.3 Loci of resonant circuits

6.3.1 Series resonant circuit

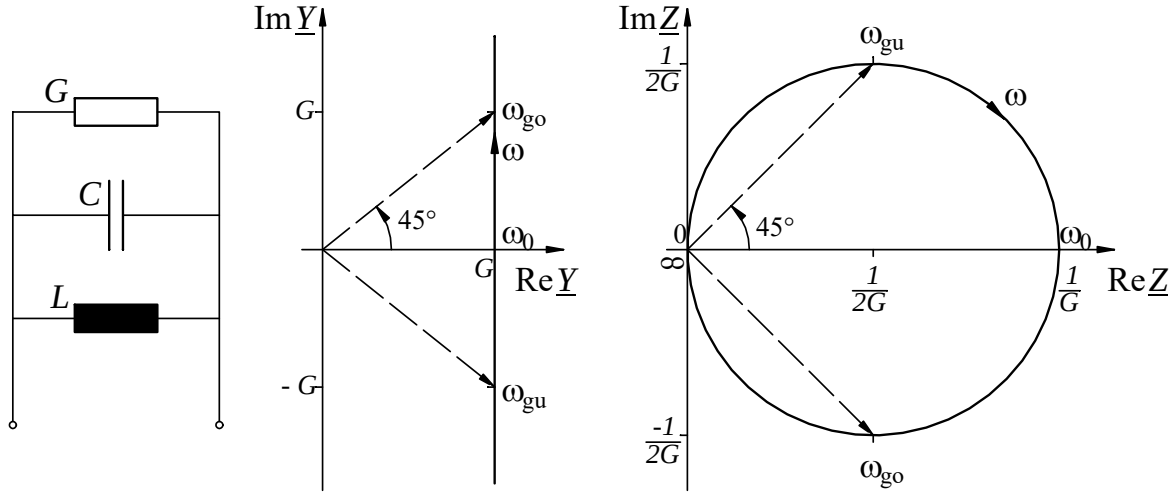
Loci of impedance and admittance of a series resonant circuit:



$$\underline{Z} = R + j\left(\omega L - \frac{1}{\omega C}\right)$$

6.3.2 Parallel resonant circuit

Loci of impedance and admittance of a parallel resonant circuit:

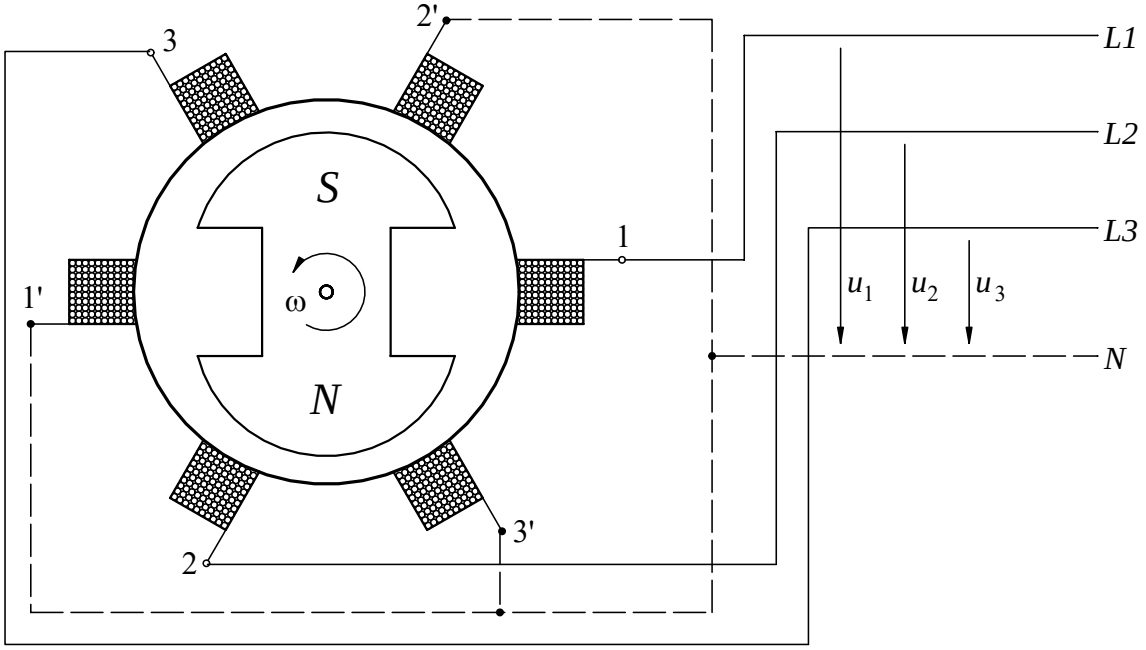


$$\underline{Y} = G + j\left(\omega C - \frac{1}{\omega L}\right)$$

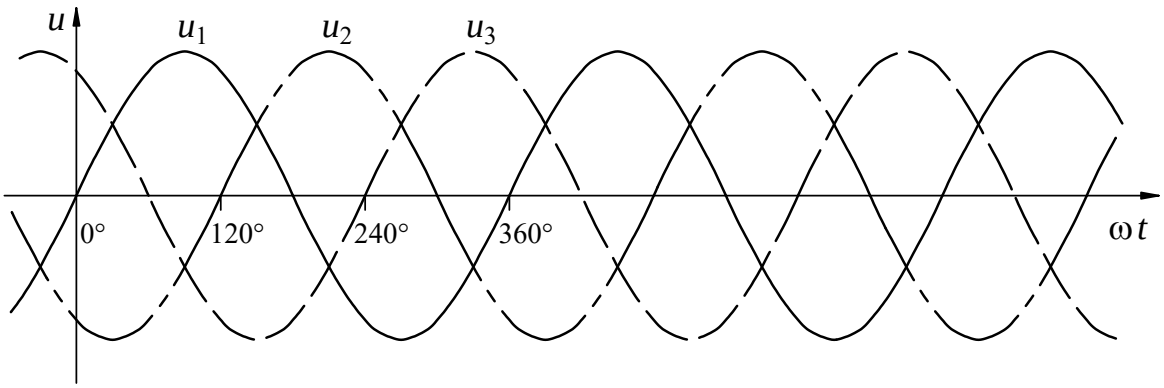
7. Three-phase systems

7.1 Generation of three-phase systems

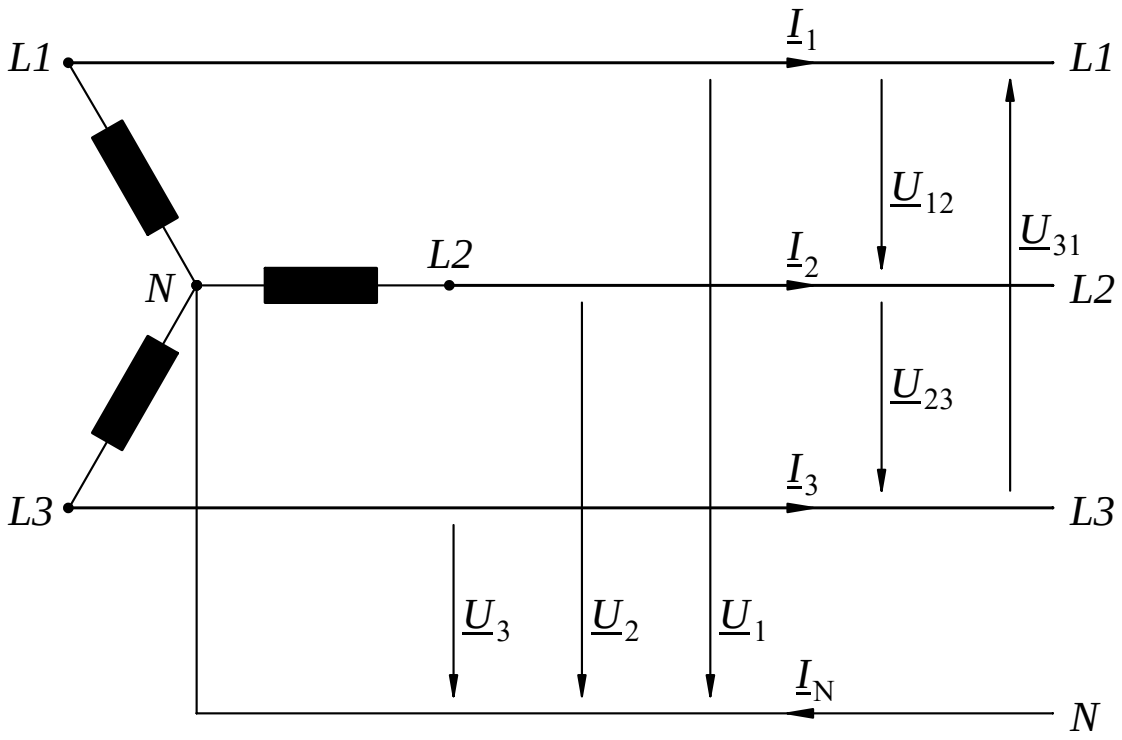
The following figure shows the cross-section of a generator with a stator that has three windings, being arranged under an angle of 120° to each other. One end of each winding is connected to each other as neutral (*N*). The other three ends are the so called phases named *L1*, *L2* and *L3*.



The rotor is a permanent magnet. Its rotation induces three equal voltages, that are shifted in phase against each other by $\alpha = \frac{2\pi}{3} = 120°$. The graph below shows the voltages induced in the three windings of the generator:



The following figure shows the equivalent circuit of the generator with the windings represented by their inductance.



With that, the different voltages can be clearly explained. The voltages between the phases and the neutral $\underline{U}_1$, $\underline{U}_2$, $\underline{U}_3$ are called star voltages while the voltages between the phases $\underline{U}_{12}$, $\underline{U}_{23}$, $\underline{U}_{31}$ are called delta voltages

Using the mesh rule, we get:

$$\underline{U}_{12} = \underline{U}_1 - \underline{U}_2$$
$$\underline{U}_{23} = \underline{U}_2 - \underline{U}_3$$
$$\underline{U}_{31} = \underline{U}_3 - \underline{U}_1$$

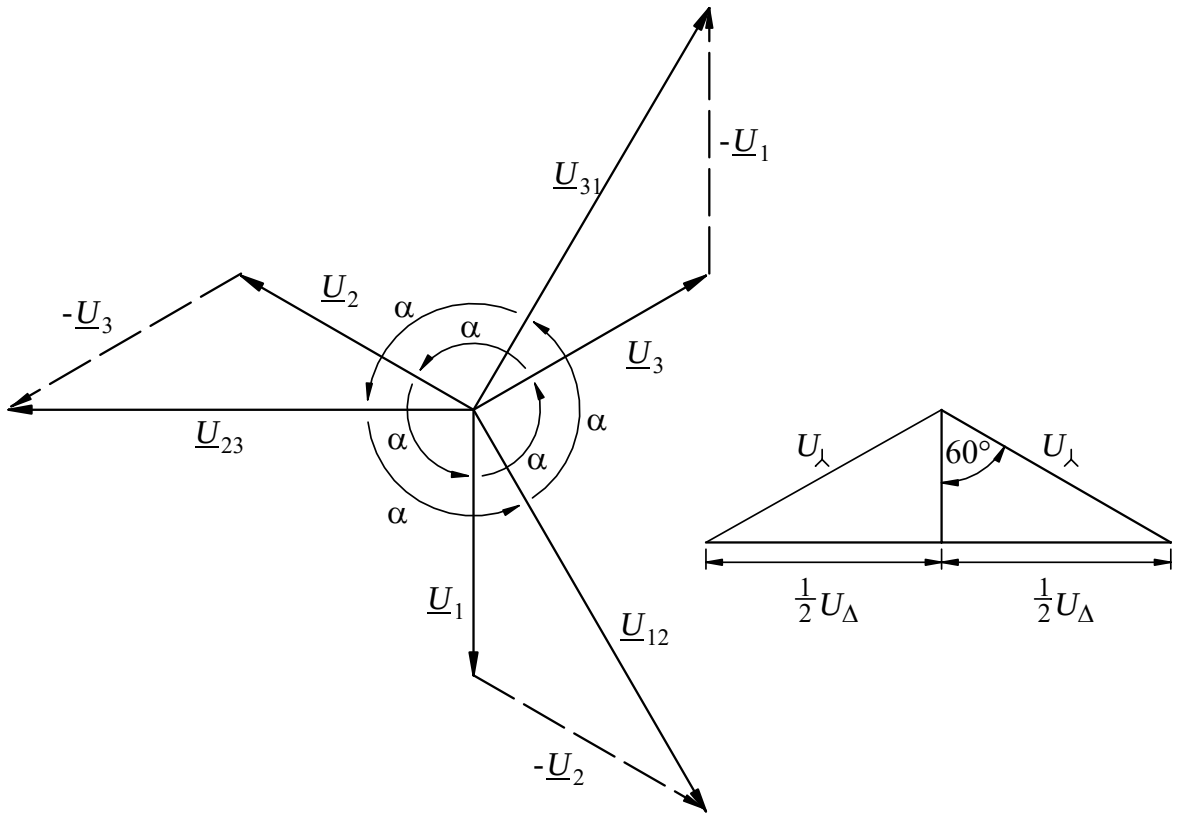
According to the node rule, the current in the neutral conductor is equal to the sum of the currents in the phase conductors:

$$\underline{I}_N = \underline{I}_1 + \underline{I}_2 + \underline{I}_3$$

The phase shift between the star voltages is:

$$\alpha = \frac{2\pi}{3} = 120°$$

Phasor diagram of the star voltages and the delta voltages:



The delta voltages as well as the star voltages have a phase angle of 120° between each other.

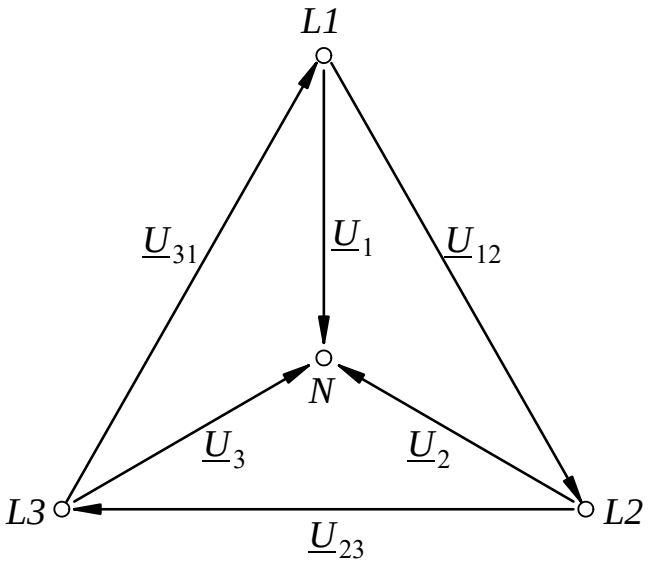
The relation between star voltage and delta voltage can be derived from the triangle shown at the right hand side:

$$\frac{\frac{1}{2}\cdot U_{\Delta}}{U_{\mathrm{Y}}} = \sin 60° = \frac{\sqrt{3}}{2}$$

$$U_{\Delta} = 2\cdot\frac{\sqrt{3}}{2}\cdot U_{\mathrm{Y}}$$

$$U_{\Delta} = \sqrt{3}\cdot U_{\mathrm{Y}}$$

Another common way to represent the voltages in a three phase system in a phasor diagram is shown below:



Phasor diagram with phases and neutral conductor

In the following the meaning of the symbols used in chapter 7 is summarized:

$\underline{I}_1, \underline{I}_2, \underline{I}_3$	line currents
$\underline{I}_N$	neutral current
$\underline{U}_1, \underline{U}_2, \underline{U}_3$	star voltages

$$U_1 = U_2 = U_3 = U_Y = \frac{U}{\sqrt{3}}$$

$\underline{U}_{12}, \underline{U}_{23}, \underline{U}_{31}$	delta voltages

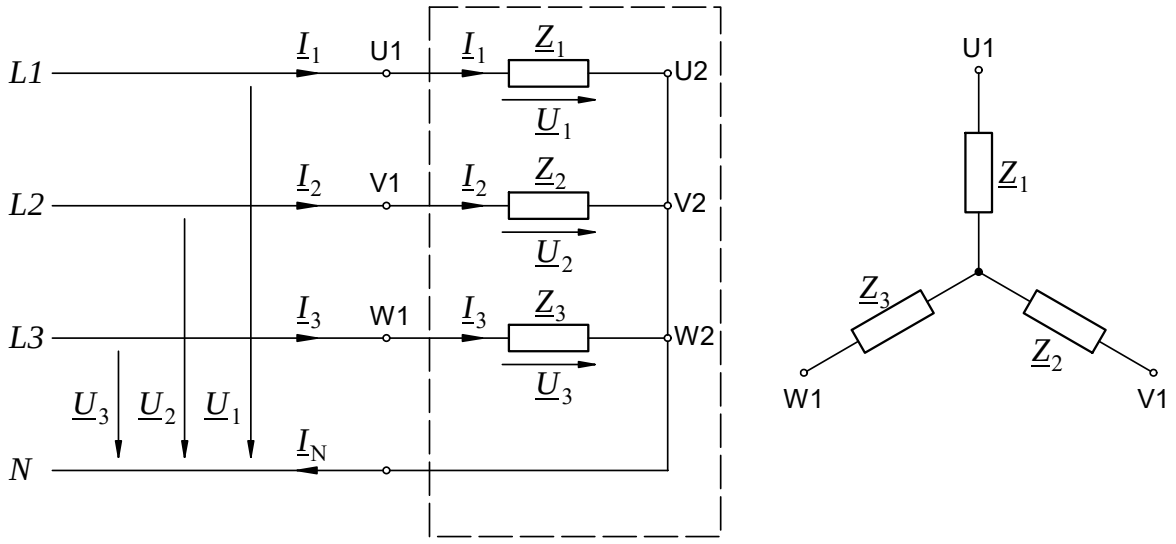
$$U_{12} = U_{23} = U_{31} = U_\Delta = U$$

U	nominal voltage of the three phase system

7.2 Balanced load

7.2.1 Star connection

Balanced load in star connection:



The letters U1, V1, W1 and U2, V2, W2 represent the terminals of the balanced load impedances.

The terms phase, phase voltage, phase current and phase resistance characterize the load.

For the star connection the following relations are to be applied:

phase voltages = star voltages $\underline{U}_1, \underline{U}_2, \underline{U}_3$

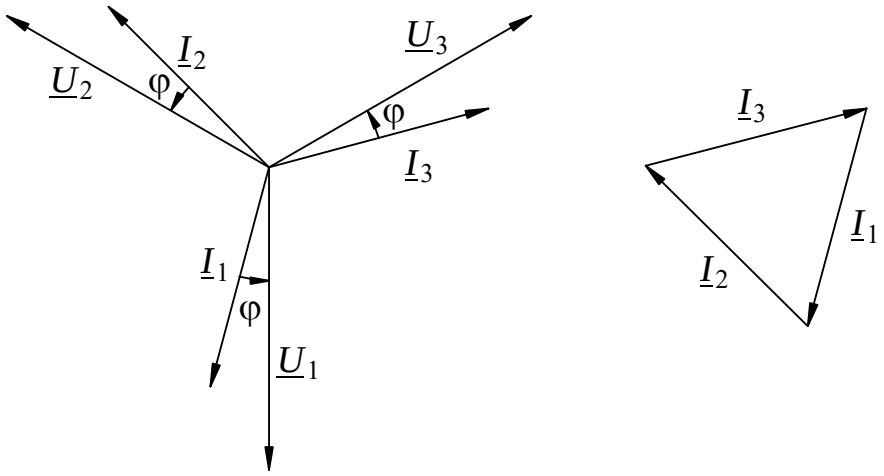
phase currents = line currents $\underline{I}_1, \underline{I}_2, \underline{I}_3$

The next figure shows the phasor diagram for a balanced inductive load with:

$$\underline{Z}_1 = \underline{Z}_2 = \underline{Z}_3 = \underline{Z} = Z \cdot e^{j\varphi}$$

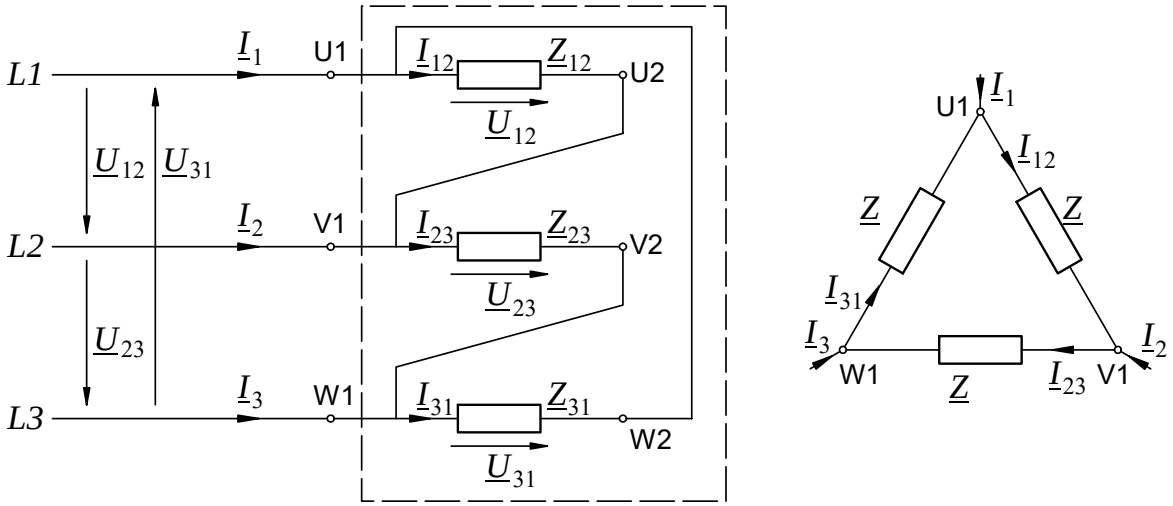
The neutral current is zero in this case:

$$\underline{I}_N = \underline{I}_1 + \underline{I}_2 + \underline{I}_3 = 0$$



7.2.2 Delta connection

Balanced load in delta connection: $Z_{12} = Z_{23} = Z_{31}$



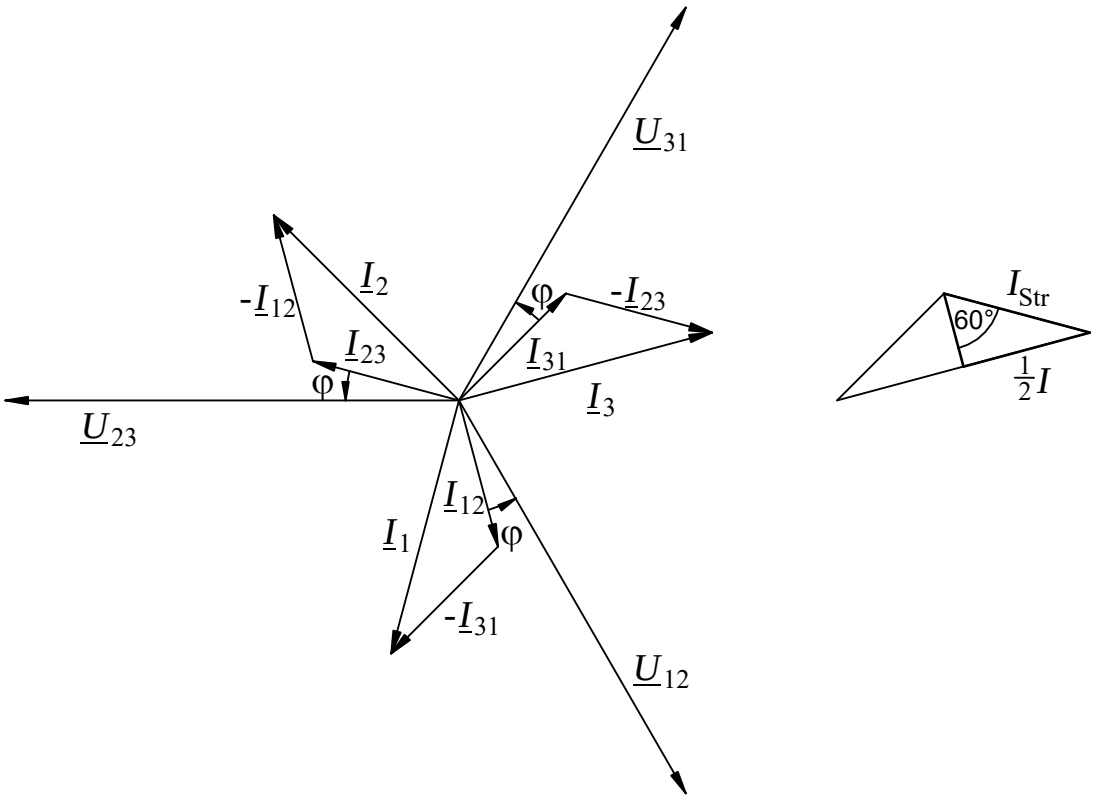
For the delta connection the following relations are to be applied:

phase voltages	= delta voltages	$\underline{U}_{12}, \underline{U}_{23}, \underline{U}_{31}$
phase currents	= delta currents	$\underline{I}_{12}, \underline{I}_{23}, \underline{I}_{31}$

The delta currents are **not** equal to the line currents. The relation between the delta currents and the line currents is:

$$\underline{I}_1 = \underline{I}_{12} - \underline{I}_{31}$$
$$\underline{I}_2 = \underline{I}_{23} - \underline{I}_{12}$$
$$\underline{I}_3 = \underline{I}_{31} - \underline{I}_{23}$$

Phasor diagram of voltages and currents at a balanced load in delta connection:



line current: $I = I_1 = I_2 = I_3$

phase current: $I_{Str} = I_{12} = I_{23} = I_{31}$

The relation between line current and phase current can be calculated (see triangle on the right hand side):

$$\frac{\frac{1}{2} \cdot I}{I_{Str}} = \sin 60° = \frac{\sqrt{3}}{2}$$

$$I = \sqrt{3} \cdot I_{Str}$$

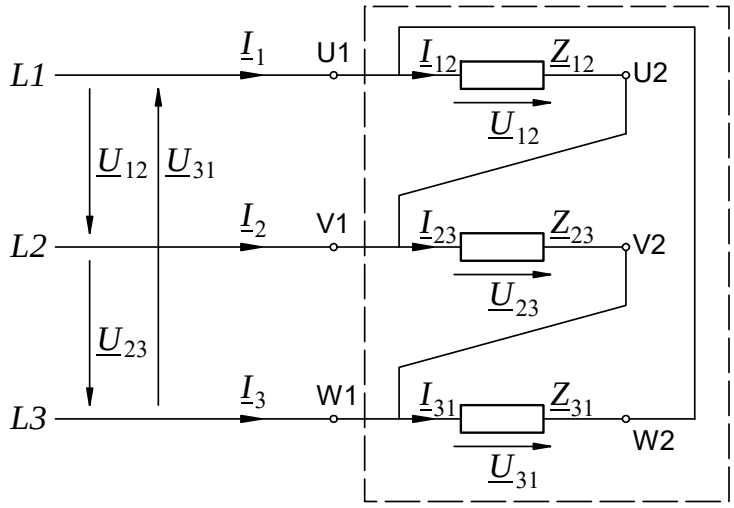
This relation is only valid for a balanced load!

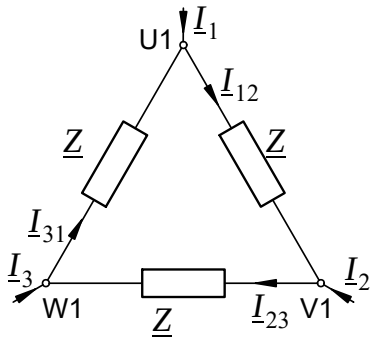
7.3 Unbalanced load

7.3.1 Delta connection

The phase voltages are equal to the delta voltages: $\underline{U}_{12}, \underline{U}_{23}, \underline{U}_{31}$.

However, the three impedances Z_{12}, Z_{23}, Z_{31} are not equal, therefore we speak about unbalanced load.





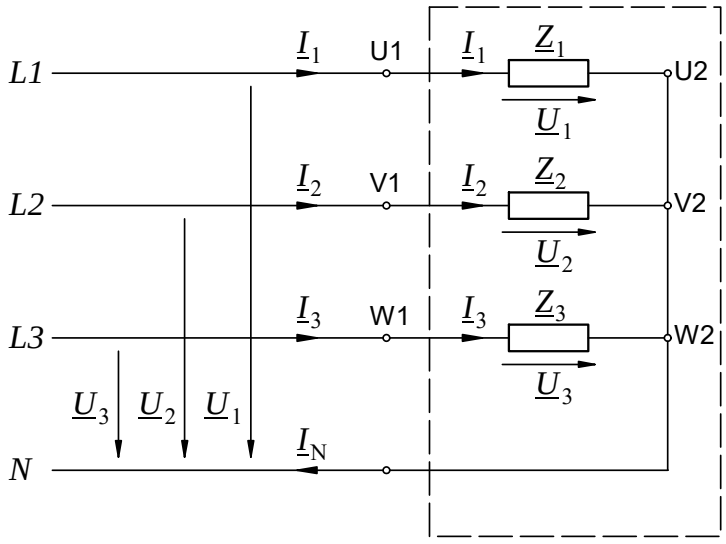
The phase currents can be easily calculated:

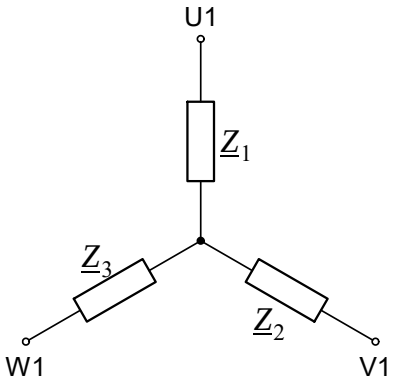
$$\underline{I}_{12} = \frac{\underline{U}_{12}}{\underline{Z}_{12}} \qquad \underline{I}_{23} = \frac{\underline{U}_{23}}{\underline{Z}_{23}} \qquad \underline{I}_{31} = \frac{\underline{U}_{31}}{\underline{Z}_{31}}$$

The line currents $\underline{I}_1, \underline{I}_2, \underline{I}_3$ can be determined by applying the node rule.

7.3.2 Star connection

a) with star point connected to the neutral:





The phase voltages are equal to the star voltages $\underline{U}_1, \underline{U}_2, \underline{U}_3$.
The phase currents can be easily calculated:

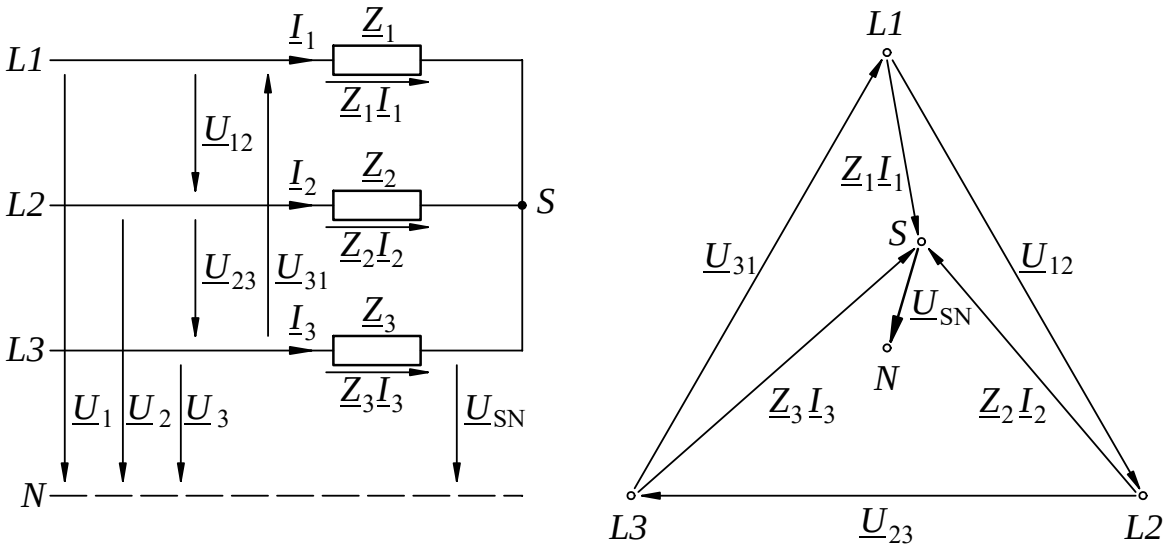
$$\underline{I}_1 = \frac{\underline{U}_1}{\underline{Z}_1} \qquad \underline{I}_2 = \frac{\underline{U}_2}{\underline{Z}_2} \qquad \underline{I}_3 = \frac{\underline{U}_3}{\underline{Z}_3}$$

The current in the neutral conductor is:

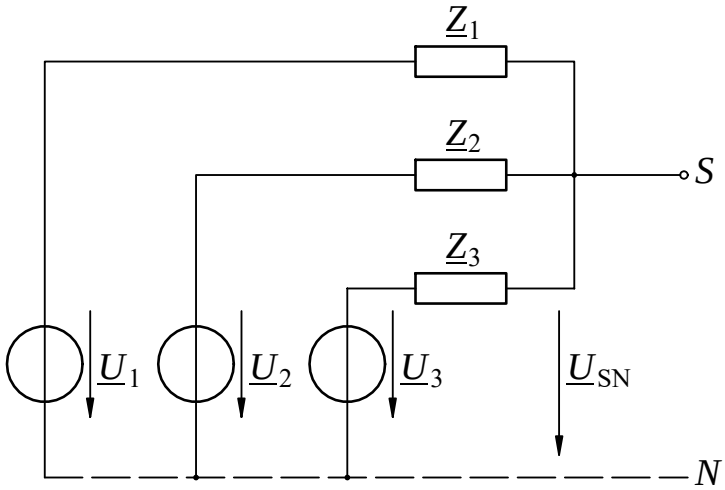
$$\underline{I}_N = \underline{I}_1 + \underline{I}_2 + \underline{I}_3$$

b) with star point **not** connected:

The following figure shows an unbalanced load in star connection with open star point:



The potential of the star point as well as the phase voltages are unknown. The potential of the star point can be determined by applying the method of the equivalent voltage source.



The short circuit current is:

$$\underline{I}_k = \frac{\underline{U}_1}{\underline{Z}_1} + \frac{\underline{U}_2}{\underline{Z}_2} + \frac{\underline{U}_3}{\underline{Z}_3}$$

The intrinsic resistance is:

$$\underline{Z}_i = \frac{1}{\frac{1}{\underline{Z}_1} + \frac{1}{\underline{Z}_2} + \frac{1}{\underline{Z}_3}}$$

The voltage between star point and neutral can then be expressed as:

$$\underline{U}_{SN} = \underline{I}_k \cdot \underline{Z}_i = \frac{\frac{\underline{U}_1}{\underline{Z}_1} + \frac{\underline{U}_2}{\underline{Z}_2} + \frac{\underline{U}_3}{\underline{Z}_3}}{\frac{1}{\underline{Z}_1} + \frac{1}{\underline{Z}_2} + \frac{1}{\underline{Z}_3}}$$

The line currents, that are equal to the phase currents, follow from that:

$$\underline{I}_1 = \frac{\underline{U}_1 - \underline{U}_{SN}}{\underline{Z}_1} \qquad \underline{I}_2 = \frac{\underline{U}_2 - \underline{U}_{SN}}{\underline{Z}_2} \qquad \underline{I}_3 = \frac{\underline{U}_3 - \underline{U}_{SN}}{\underline{Z}_3}$$

The following table gives some general rules for the determination of phase voltage and phase current in balanced and unbalanced three phase systems:

	phase voltage	phase current
star connection	$U_{Str} = U_{\mathrm{Y}} = \frac{U}{\sqrt{3}}$ always for connected star point and balanced load	$I_{Str} = I$ for balanced load, otherwise different for each phase
delta connection	$U_{Str} = U_{\Delta} = U$ always	$I_{Str} = \frac{I}{\sqrt{3}}$ for balanced load, otherwise different for each phase

7.4 Power in three-phase systems

7.4.1 Active, reactive and apparent power

In chapter 5.7.5 the following relations for power in AC-systems were derived:

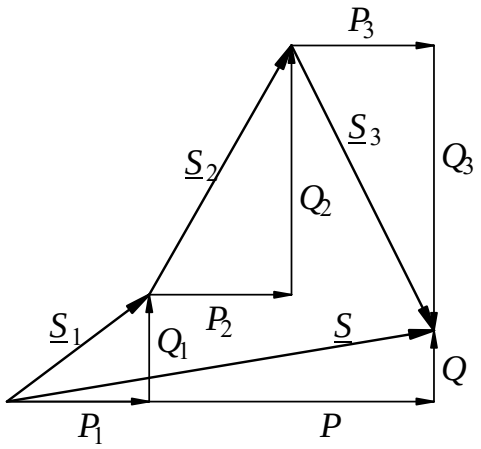
$$\underline{S} = \underline{U} \cdot \underline{I}^{*} = U \cdot e^{j\varphi_u} \cdot I \cdot e^{-j\varphi_i} = U \cdot I \cdot e^{j\varphi} = U \cdot I \cdot \cos\varphi + jU \cdot I \cdot \sin\varphi$$
$$P = \mathrm{Re}\,\underline{S} = S \cdot \cos\varphi = U \cdot I \cdot \cos\varphi$$
$$Q = \mathrm{Im}\,\underline{S} = S \cdot \sin\varphi = U \cdot I \cdot \sin\varphi$$

In a three phase system the total power is the sum of the powers in the three phases. Active power and reactive power can be added arithmetically while the apparent power has to be added geometrically:

$$P = P_1 + P_2 + P_3$$

$$Q = Q_1 + Q_2 + Q_3$$

$$\underline{S} = \underline{S}_1 + \underline{S}_2 + \underline{S}_3$$



For the special case of a balanced load at a symmetrical power supply all phase powers are equal and the angle/direction of the apparent power phasors is the same. So the total power is simply three times the phase power:

$$P = 3 \cdot P_{Str} = 3 \cdot U_{Str} \cdot I_{Str} \cdot \cos\varphi$$
$$Q = 3 \cdot Q_{Str} = 3 \cdot U_{Str} \cdot I_{Str} \cdot \sin\varphi$$
$$S = 3 \cdot S_{Str} = 3 \cdot U_{Str} \cdot I_{Str}$$

For star and delta connection this results in an apparent power:

star	$U_{Str} = U_{Y} = \frac{U}{\sqrt{3}}$ $I_{Str} = I_{Y} = I$	$S = 3 \cdot \frac{U}{\sqrt{3}} \cdot I = \sqrt{3} \cdot U \cdot I$
delta	$U_{Str} = U_{\Delta} = U$ $I_{Str} = I_{\Delta} = \frac{I}{\sqrt{3}}$	$S = 3 \cdot U \cdot \frac{I}{\sqrt{3}} = \sqrt{3} \cdot U \cdot I$

The same relations are valid for active and reactive power, regardless of the circuit type:

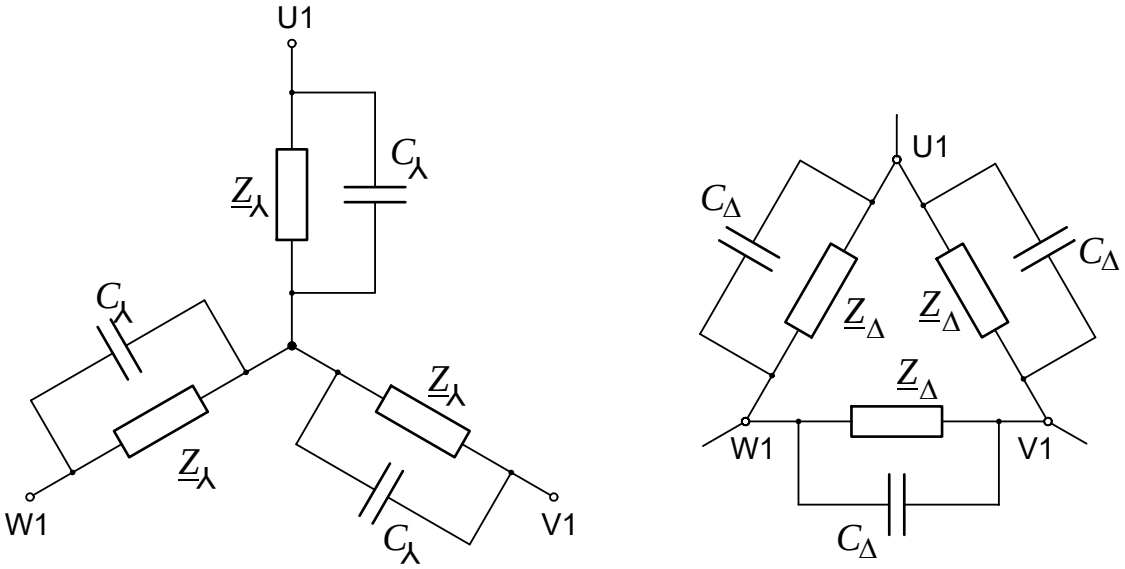
$$S = \sqrt{3} \cdot U \cdot I$$
$$P = \sqrt{3} \cdot U \cdot I \cdot \cos\varphi$$
$$Q = \sqrt{3} \cdot U \cdot I \cdot \sin\varphi$$

Power is generally given in that form with *U* being the nominal voltage of the power system. U can always be measured. U_Y can only be measured if there is access to the neutral conductor.

7.4.2 Reactive power compensation

Reactive power compensation in three phase systems follows the same principles as described in chapter 5.7.6 for one phase loads. Here only the case of a balanced load will be discussed. In case of an unbalanced load the compensation elements are not equal for all three phases; therefore they have to be calculated separately.

Reactive power compensation for a balanced, inductive consumer in delta and star connection:



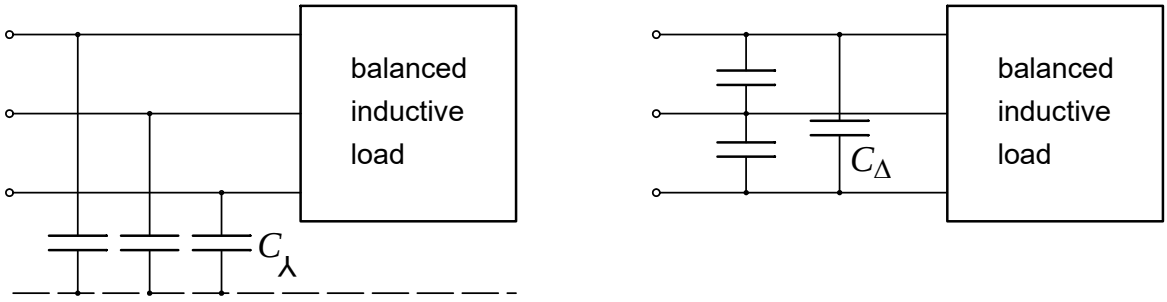
The necessary capacitance can be determined by:

$$Q_{Str} = \omega \cdot C_{Str} \cdot U_{Str}^2$$
$$C_{Str} = \frac{Q_{Str}}{\omega \cdot U_{Str}^2} = \frac{Q}{3 \cdot \omega \cdot U_{Str}^2}$$

star	delta
$U_{Str} = U_{\mathrm{Y}} = \frac{U}{\sqrt{3}}$	$U_{Str} = U_{\Delta} = U$
$C_{\mathrm{Y}} = \frac{Q}{3 \cdot \omega \cdot \left(\frac{U}{\sqrt{3}}\right)^2} = \frac{Q}{\omega \cdot U^2}$	$C_{\Delta} = \frac{Q}{3 \cdot \omega \cdot U^2}$

That shows, that $C_{\mathrm{Y}} = 3 \cdot C_{\Delta}$, but the voltage at C_{Δ} is higher by a factor of $\sqrt{3}$.

Star and delta connection for reactive power compensation:



For the design of a reactive power compensation for a balanced load the following rules can be used:

important:	the reactive power to be compensated
not important:	he type of load circuit (star or delta)
usual:	delta connection, because the capacitances needed are smaller by a factor of 3

8. Protection and safety measures

Mandatory requirements with a character similar to a law: Standard series DIN VDE 0100

8.1 Fundamentals

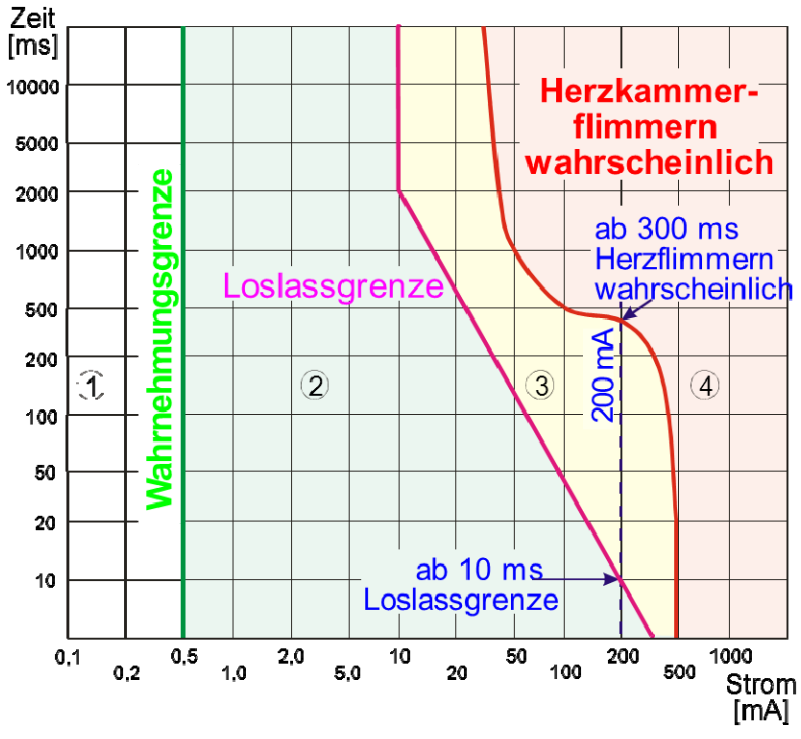
We have to distinguish between the protection against direct contacts and the protection against indirect contacts. The latter is necessary, if non-active, but electrically conductive parts of an apparatus receive dangerous touch voltages due to failures. The possible protection measures depend on the applied electrical network configuration. We differ in:

TN: TN-S, TN-C, TN-C-S
TT
IT

The 1st letter characterizes the treatment of the network's starpoint, the 2nd letter the treatment of the apparatus' chassis: T is earth (terra), N is neutral conductor and I is isolation. A consecutive S for the network configuration TN indicates, that protective earth (PE) and neutral conductor (N) are separated in the installation, a consecutive N indicates, that these are combined in one conductor.

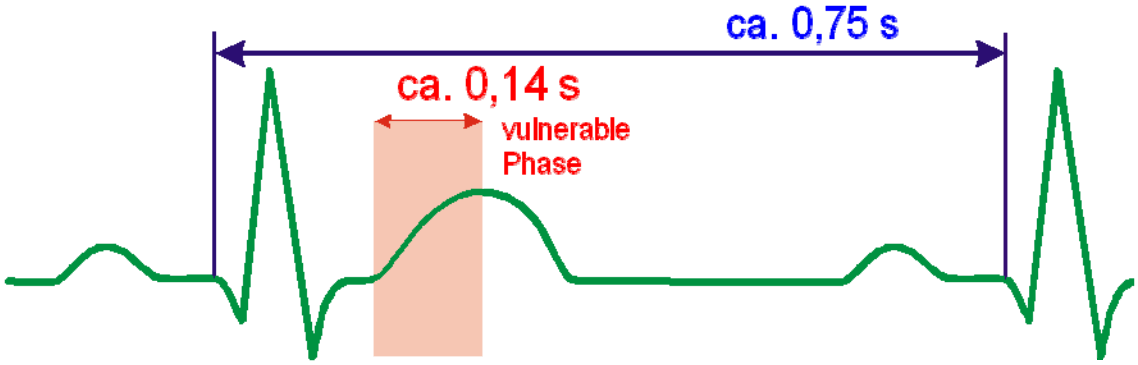
8.2 Effects of the current on the human body

The reactions of a current on the human body depend on the exposure time and the current value through the body. Based on investigations four areas or zones can be distinguished [DIN IEC 60479-1]:

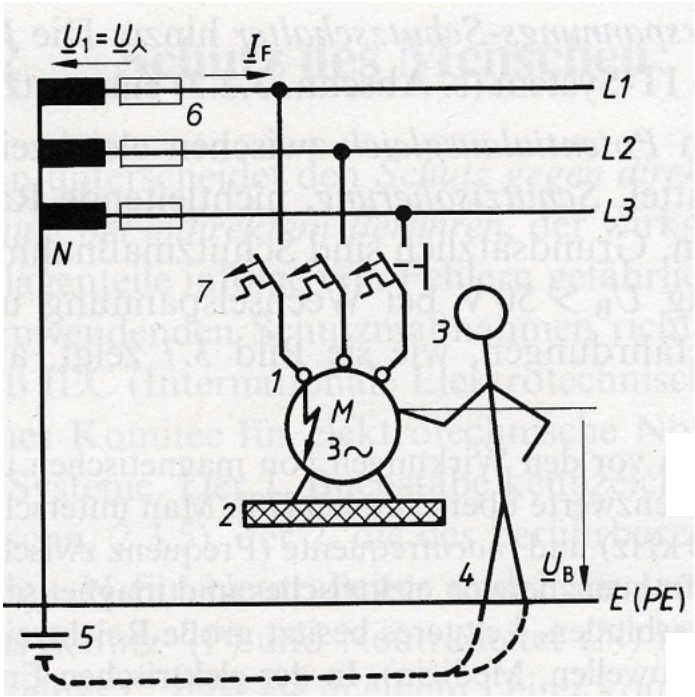


Zone 1: no effects and reactions.
Zone 2: no harmful physiological effects.
Zone 3: no organ defects expected, reversible disfunction of the generation and the conduction of the heart's stimulus, anaerobic muscle contraction and breathing problems probable.
Zone 4: ventricular fibrillation probable, increasing current value and exposure time lead to cardiac arrest, breathing arrest and heavy burnings. Above 500 mA each short exposure may be deadly.

In case of an AC current through the human heart, it attempts to follow these quicker and larger external impulses. Therefore it beats quicker. Heart rhythm disturbances may occur, i.e. the heart beats abnormally. If the external current impulse falls in the so-called vulnerable phase, ventricular fibrillation occurs. As a consequence, a cardiac arrest occurs, followed by a circulation arrest. This leads to death [DIN IEC 60479-1].



If we assume the electrical resistance of a human body to be R = 1 kΩ and the body current to be I = 50 mA, the tolerable touch voltage is U = 50 V.



1 three-phase motor
2 mounting plate
3 person
4 position
5 operational earth
6 fuses
7 motor circuit breaker

The previous figure shows an accidental current circuit. Due to the faulty connection of the phase conductor L1 to the chassis of the three-phase motor the person gets the touch voltage U_B against its position. To evaluate the danger we use U_B. However, more important is the current through the body.

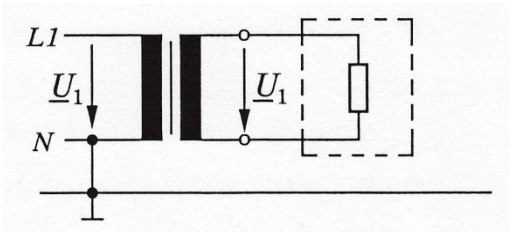
8.3 Protection measures [DIN VDE 0100 Teil 410]

8.3.1 Safety extra-low voltage

The electrical circuits are operated at a voltage below the tolerable touch voltage of 50 V AC (120 V in case of DC). Therefore no dangerous touch voltages are possible (except voltage resonances occur, see chapter 5.6.6).

8.3.2 Protective separation

An isolating transformer is connected between the supply network and the consumer. With the isolating transformer voltages from the supply network are avioded. At the output, no voltages against the earth exist.



To ensure this protection measure, the isolating transformer must be connected to one consumer only. Mobile isolating transformers must have protective isolation.

8.3.3 Protective isolation

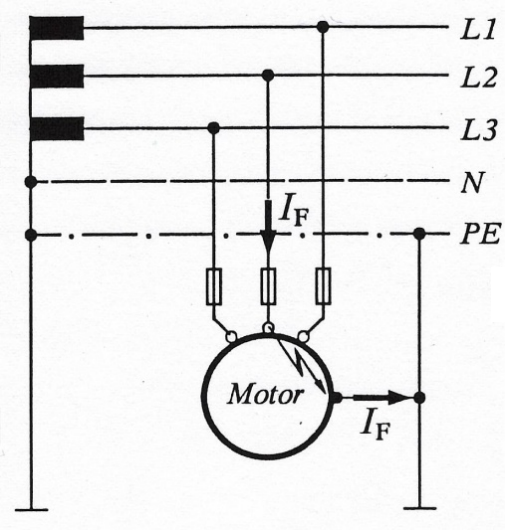
This is the usually used protection against electric shock: the protection against direct contacts. It is achieved by isolating all active parts of a circuit with coverings, coatings, envelopes, enclosures, sheaths, e.g.:

- Laboratory conductors with silicon coatings,
- Plastic cases of a drilling machine and a coffee maker,
- Plastic covers for electrical sockets and light switches.

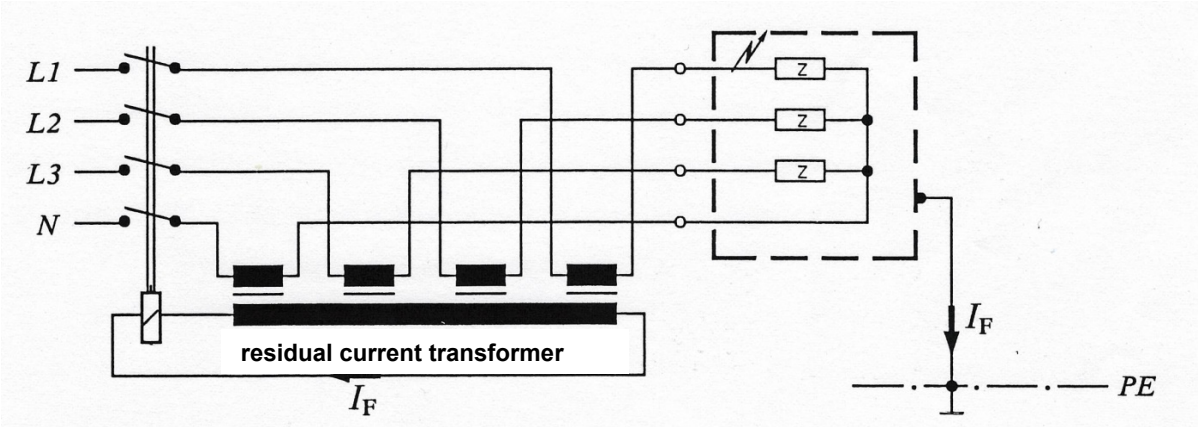
8.3.4 Protective earthing

Here the protective earth (PE) is connected with the body (chassis) of the electrical apparatus. This protection measure is system dependent. The PE is always marked as green & yellow.

The fuse immediately switches off the failure current.



8.3.5 Residual current circuit breaker (RCCB)



The residual current protective device (RCD) measures the sum of the currents using a summation current transformer. In the usual operation mode (no failure) this sum must be zero. In case of a faulty connection of a line conductor to the earthed body (chassis) the fault current I_F flows over the earth back to the star point of the transformer. This current now also occurs at the secondary side of the summation current transformer, so that the current relay is activated (e.g. I_F > 30 mA).

8.3.6 Emergency stop button

In laboratories and especially in eductional rooms a mechanism for an emergency stop must be installed. If this button is pushed in case of an emergency, the power supply for all tables and electrical facilities of the room is interrupted (power-off, currentless).

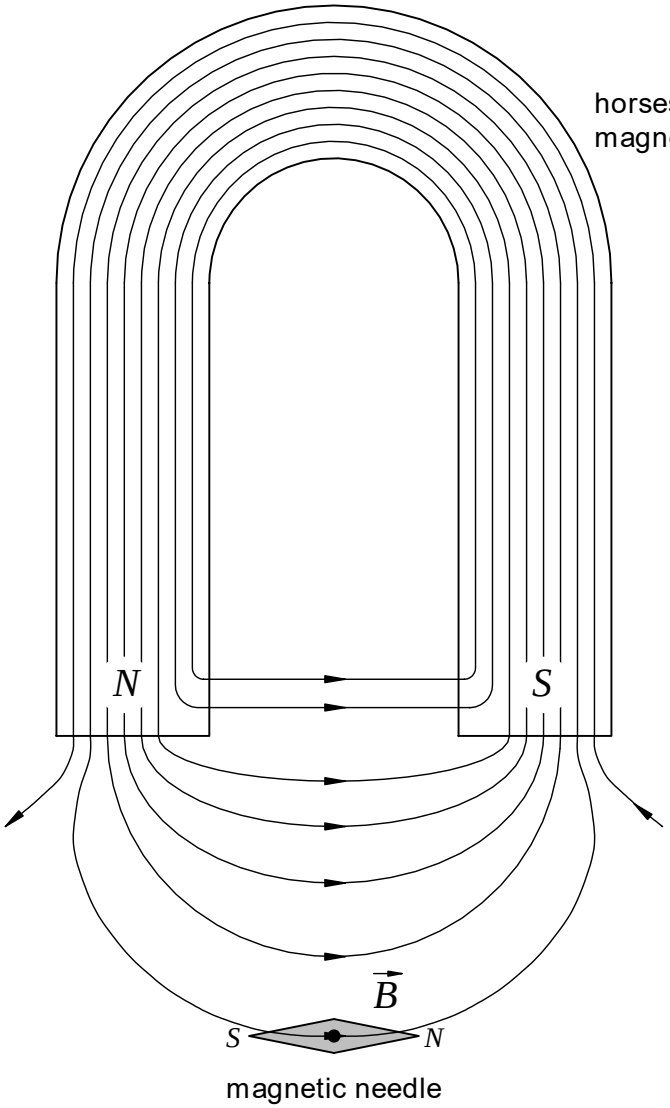
The switch for reactivating the power supply after the emergency stop must be secured against unauthorized use and may be used only by the authorized persons after the electrical fault is removed and eliminated.

9. Time-independent magnetic fields

9.1 Introduction / Nature of magnetism

9.1.1 Polarity and field lines

In this section we will deal with the cause and description of the magnetic field. In addition, characteristic field quantities are defined and a cause is assigned to them, as with the electric field. Also like the electric field, these quantities are dependent on the environment in which the magnetic field expands. We will therefore also deal with the material dependency of the magnetic field. First, however, it is a matter of describing the magnetic field in a material-free space.



A permanent magnet exerts a directivity on a magnet needle.

Description: → Vector field, Magnetic field $\vec{B}$

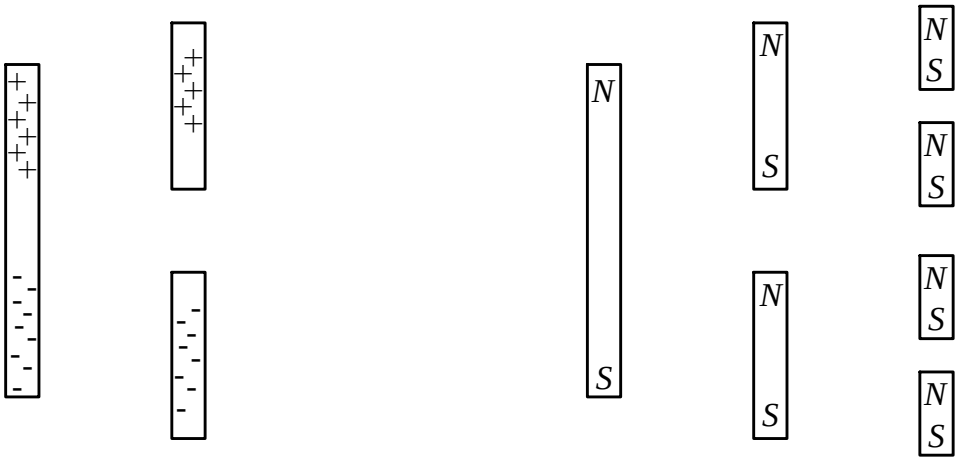
Quantity = Intensity of directive efficiency

Direction = Direction of the magnet needle

North pole: = the pole of a bar magnet showing to the north when it is freely suspended.

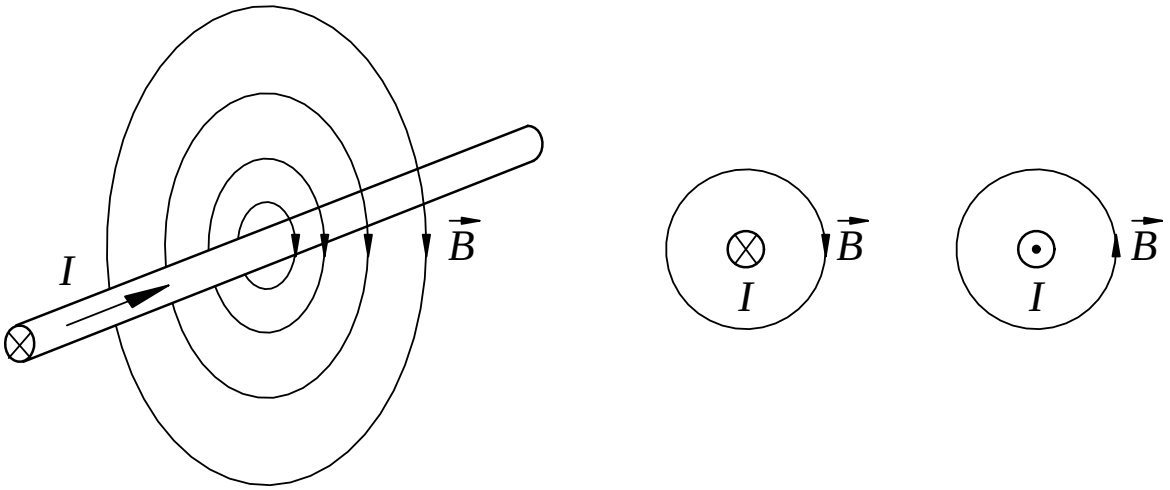
Field lines
exit the magnet at the north pole and re-enter at the south pole.
They close over the magnets.

There are no magnetic charges!

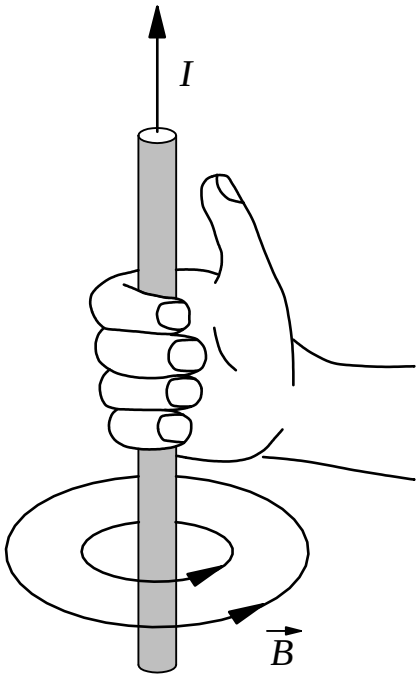


electric magnetic

Electrified conductors exert a directive efficiency on the magnet needles.



- Closed loop field lines
- intensity decreases with $\frac{1}{r} \Rightarrow B \sim \frac{1}{r}$
- clockwise winding correlation of electric current and magnetic field (right hand rule)



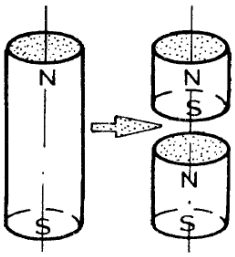
The direction of the magnetic field lines, due to the direction of the current, can be investigated using the so-called right hand rule. If the thumb of the right hand shows in current direction, the other fingers show in the magnetic field lines direction [4].

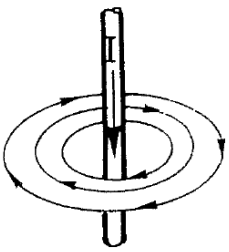
Another method to save this correlation is the conception of a screw with right-handed thread. The current then is represented by the axial movement of the screw while the magnetic field lines direction is given by the screw's rotating direction.

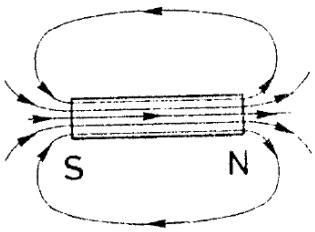
As will be shown later, the field lines around an infinitely long, straight conductor form concentric circles around it.

We therefore know two causes for a magnetic field: a magnetic field can be created by a so-called permanent magnet or by an electric current, which means by moving charges.

In comparison to the electric field there is a fundamental difference: The electric field is caused by charges, which can be divided into positive and negative charges. In contrast, the poles of a magnet never occur individually. In an elementary magnet, a north and a south pole always exist together. As a result, the field lines in a magnetic field don't have a starting or an ending point, but are always a closed loop.

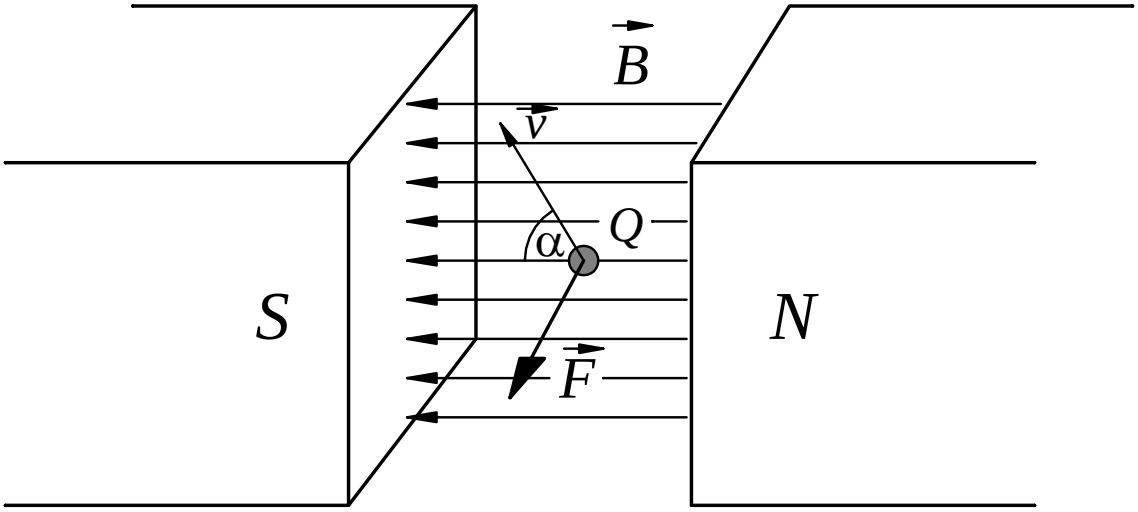




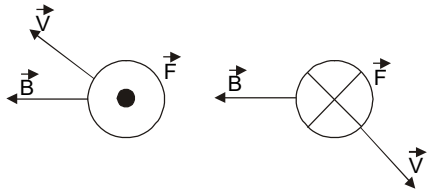


9.1.2 Force on a moving electric charge

The force between two stationary electric charges can be calculated using Coulomb's law. This force, although small, can be verified by an experiment. If these two charges move, an additional force between them can be verified. This additional force is referred to as the magnetic force, the so-called Lorentz force.



The experiment results in: $F \sim Q \cdot v \cdot B \cdot \sin(\alpha)$

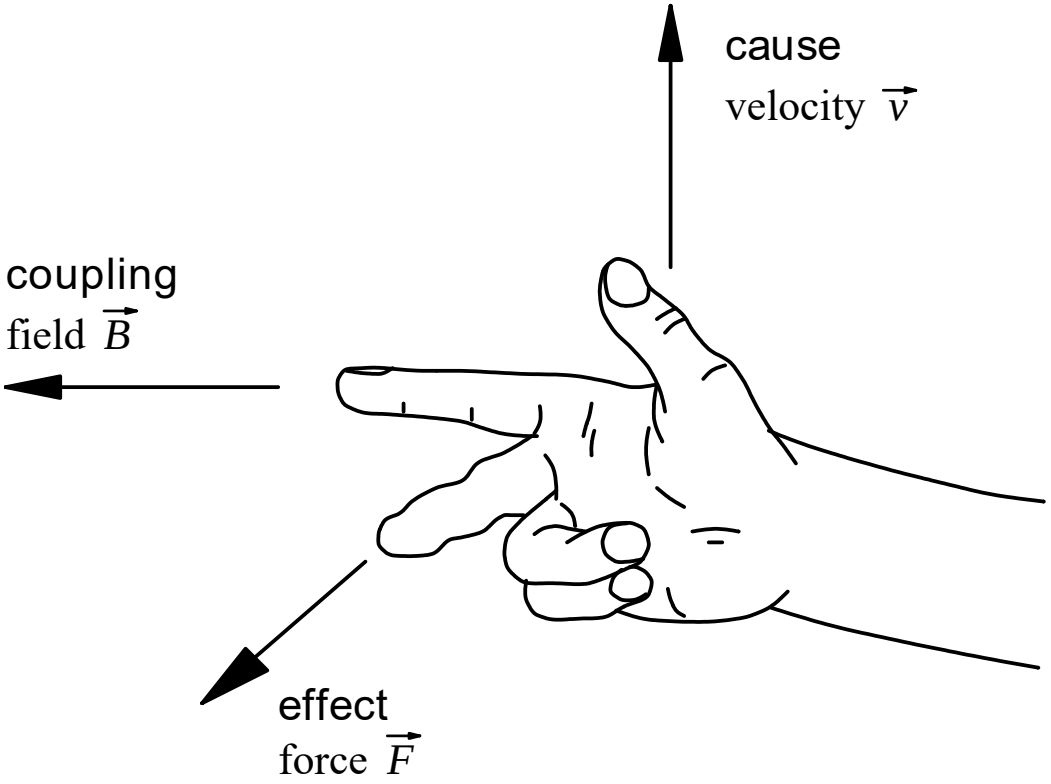


Force direction $\vec{F} \cong$ Feed direction of a right hand helix, when the Vector $\vec{v}$ is turned into the Vector $\vec{B}$ by the smallest angle (for Q>0!)

Mathematical Description:

$$\boxed{\vec{F} = Q \cdot \vec{v} \times \vec{B}}$$

The quantity $\vec{B}$ is defined through the choice of the proportionality constant = 1. (compare section 9.1.5)



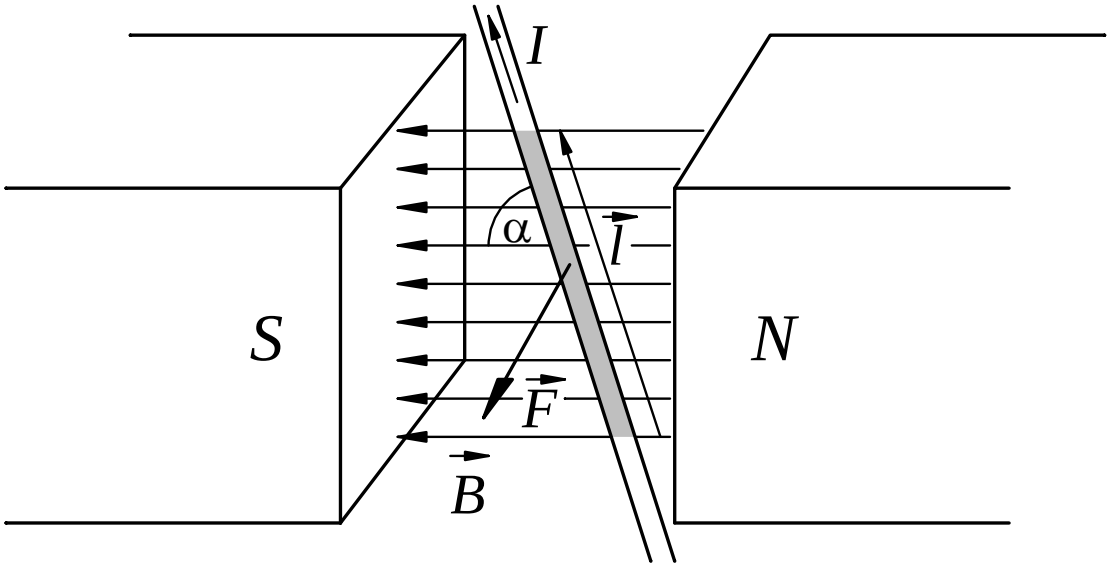
Easily portrayable as the U-V-W rule of the right hand [4]:

- cause (U): Movement v (Current I)
- coupling (V): Magnetic field B
- effect (W): Force F.

Note: The Hall-Effect portrays a technical application of the force effect on moving electric charges.

9.1.3 Force on a current-carrying conductor

The magnetic force between individual moving electric charges is very small compared to the Coulomb force, so that the magnetic force between two moving electric charges can hardly be verified experimentally. However, if a multitude of charges are moved, as in a current in a metal conductor, then easily measurable forces result. These forces are so large that they are applied in a variety of systems. The entire electrical machine technology is based upon the application of magnetic forces.



t is the time which the charge needs to travel the distance l.

$$I = \frac{Q}{t} \qquad Q = I \cdot t \qquad \vec{v} = \frac{\vec{l}}{t} \qquad \vec{l} = \text{vector in current direction}$$

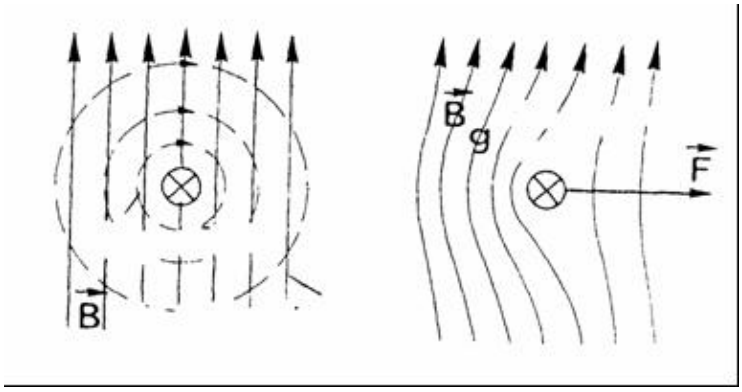
$$\vec{F} = Q \cdot \vec{v} \times \vec{B}$$

$$\vec{F} = I \cdot t \cdot \frac{\vec{l}}{t} \times \vec{B}$$

$$\boxed{\vec{F} = I \cdot \vec{l} \times \vec{B}}$$

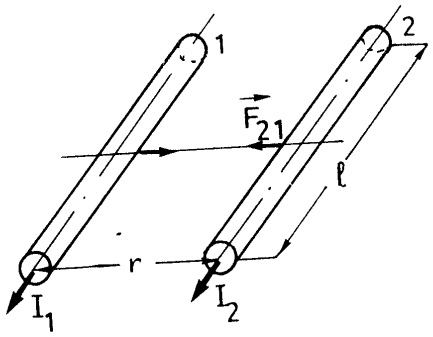
$$F = I \cdot l \cdot B \cdot \sin(\alpha)$$

graphic illustration of the force effect:



9.1.4 Force between two conductors

In the neighbouring picture, two metal conductors are arranged in parallel. They conduct the currents I_1 and I_2. The distance between the two conductors is r [19.



Through an experiment it can be proven that a force exists between the two conductors.

$$F = k \cdot \frac{I_1 \cdot I_2 \cdot l}{r}$$

The proportionality constant k is equal to:

$$k = \frac{\mu_0}{2\pi} \qquad \text{and the result is:} \qquad F = \frac{\mu_0 \cdot I_1 \cdot I_2 \cdot l}{2\pi \cdot r}$$

The dimension of the physical constant μ_0 is given by the dimension of force, the dimension of length, and the dimension of current:

$$[\mu_0] = \frac{[F] \cdot [l]}{[l] \cdot [I]^2} = \frac{VAs \cdot m}{m \cdot m \cdot A^2} = \frac{Vs}{Am}$$

This physical constant is identified as the permeability of free space or the magnetic constant. Its value is (compare section 9.1.7).

$$\mu_0 = 4\pi \cdot 10^{-7} \frac{Vs}{Am}$$

The numerical value for μ_0 results also from the definition of the current - unit:
When two parallel conductors in a vacuum with d = 20 cm spacing per meter of length exert a force F = 1 N on each other, the electric current is defined as 1 kA.

Incidentally, the two conductors attract each other when both currents flow in the same direction (as depicted in the picture); they repel each other when the two currents flow in opposing directions.

9.1.5 Magnetic flux density, Induction

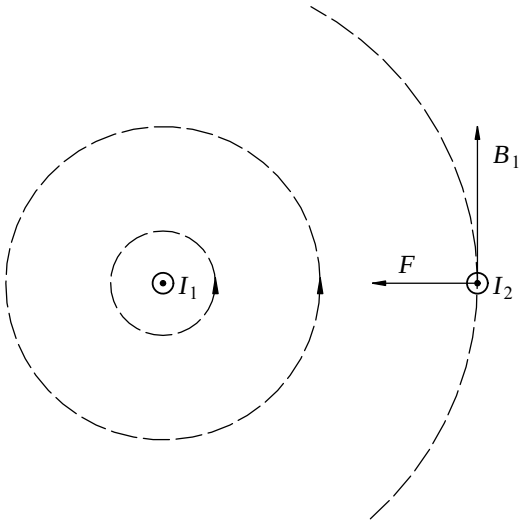
$\vec{B}$ describes the effect of the magnetic field. This quantity is termed magnetic induction or magnetic flux density.

$\vec{B}$ is defined because in the proportional context $\vec{F} \sim Q \cdot \vec{v} \times \vec{B}$, the constant of proportionality = 1 is chosen.

$$\vec{F} = Q \cdot \vec{v} \times \vec{B}$$

$$\text{Unit [B]} = \frac{N}{As \cdot \frac{m}{s}} = \frac{Nm}{Am^2} = \frac{Ws}{Am^2} = \frac{Vs}{m^2} = T \text{ (Tesla)}$$

9.1.6 Magnetic field strength



A long electrified conductor creates a magnetic field which decreases with 1/r (compare 9.1.1).
At the same time, the induction $\vec{B}$ is also dependent on the material:

$$B = \frac{\mu}{2 \cdot \pi} \cdot \frac{I}{r}$$

×

μ = Material constant (see 9.1.7)

$\frac{1}{2 \cdot \pi}$ = fixed proportionality constant (see 9.2.1)

The effect of the magnetic field is not only dependent on its cause and location, but also on the material.

Definition of a second field quantity which only correlates to the cause of the field, but ignores the material characteristics:

Magnetic field strength $H = \frac{1}{2 \cdot \pi} \cdot \frac{I}{r}$ $\qquad$ $\text{Unit}[H] = \frac{A}{m}$

The equations mentioned before are valid for the magnetic field of the long current carrying conductor.

In general, it is deemed: $\boxed{\vec{B} = \mu \cdot \vec{H}}$

9.1.7 Permeability

$$\mu = \frac{B}{H} \qquad \text{Unit}[\mu] = \frac{\frac{T}{A}}{\frac{m}{m}} = \frac{\frac{\frac{Vs}{m^2}}{A}}{\frac{m}{m}} = \frac{Vs}{Am}$$

$$\mu = \mu_r \cdot \mu_0$$

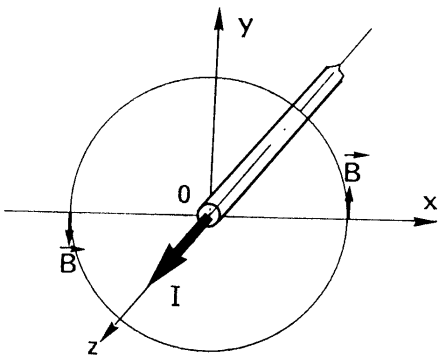
Magnetic field constant $\mu_0 = 4\pi \cdot 10^{-7} \frac{Vs}{Am} = 1{,}257 \cdot 10^{-6} \frac{Vs}{Am}$

(permeability of free space)

Relative permeability μ_r

9.1.8 Magnetic flux density around long, straight conductors

We now examine the magnetic flux density around an infinitely long, straight current-carrying conductor. This conductor is regimented into a Cartesian coordinate system. The conductor extends along the z - axis. To calculate the magnetic flux density along the x - axis, we can describe the defining equation of the magnetic flux density B (see 9.1.6) as follows:



left side

$$\vec{B}(x) = -\vec{e}_y \cdot \frac{\mu_0 \cdot I}{2\pi \cdot |x|}$$

right side

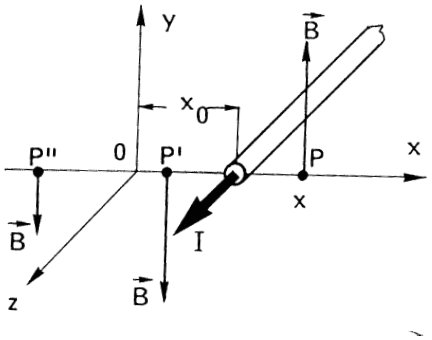
$$\vec{B}(x) = \vec{e}_y \cdot \frac{\mu_0 \cdot I}{2\pi \cdot |x|}$$

Combined in one form yields:

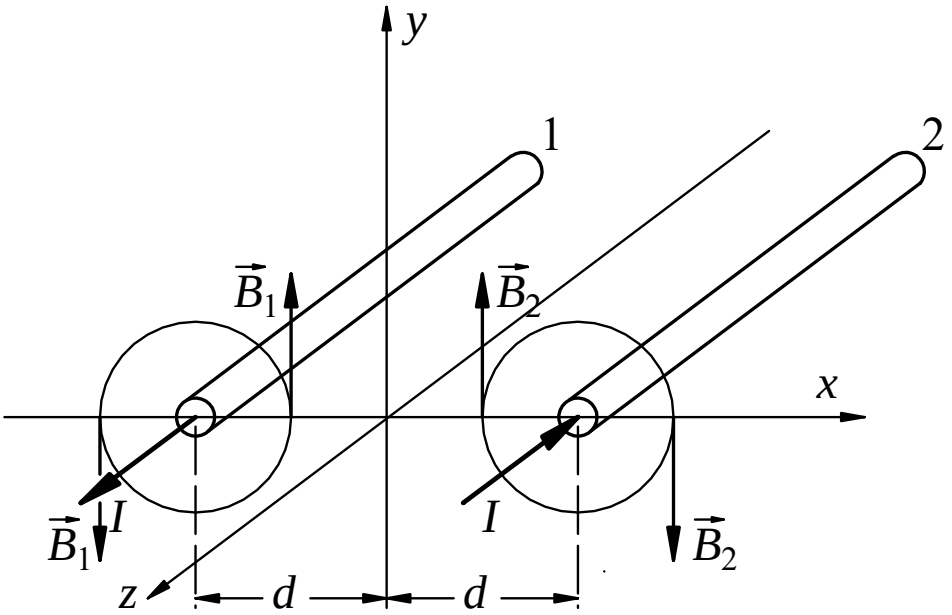
$$\vec{B}(x) = \vec{e}_y \cdot \frac{\mu_0 \cdot I}{2\pi \cdot x}$$

In many cases, the conductor is not located in the center of the coordinate system. In this case, the formula for shifting the conductor away from the origin must be adapted:

$$\vec{B}(x) = \frac{\mu_0 \cdot I}{2\pi \cdot (x - x_0)} \cdot \vec{e}_y$$



Now the arrangement of the two parallel conductors can be approached as has been described in 9.1.4. However, the currents in both conductors now flow in opposing directions. The two conductors are on the plane defined by the x - and z - axes and are now parallel to the z - axis. The distance from the origin of the coordinate system is assigned the value of +/-d.



We can now formulate the field along the x - axis for each conductor. For the conductor left of the origin it is deemed: and for the right conductor it is deemed:

$$\vec{B}_1 = \vec{e}_y \cdot \frac{\mu_0 \cdot I}{2\pi \cdot (x + d)}$$

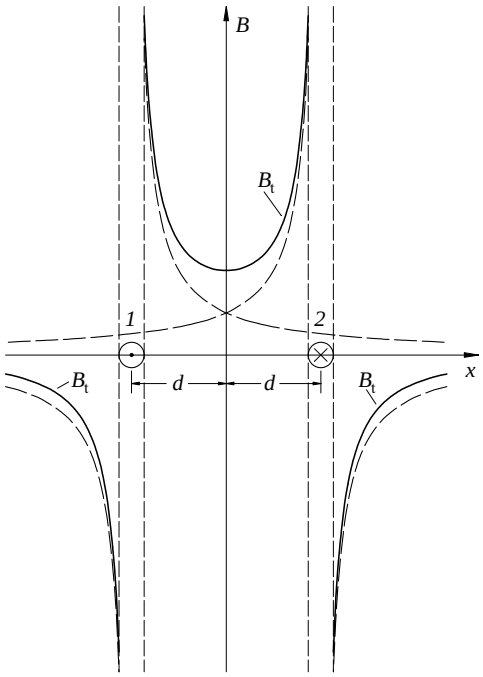
$$\vec{B}_2 = \vec{e}_y \cdot \frac{\mu_0 \cdot (-I)}{2\pi \cdot (x - d)}$$

The whole field results from the superposition of the two individual fields:

$$\vec{B} = \vec{B}_1 + \vec{B}_2 = \vec{e}_y \cdot \frac{\mu_0 \cdot I}{2\pi} \cdot \left(\frac{1}{(d + x)} + \frac{1}{(d - x)} \right)$$

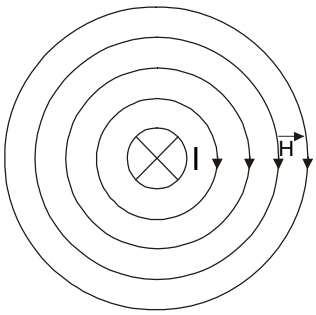
$$\vec{B} = \vec{e}_y \cdot \frac{\mu_0 \cdot I \cdot d}{\pi \cdot (d^2 - x^2)}$$

The result is the distribution of the
magnetic flux density through
a conducting loop (see picture).



9.2 Ampere's law

9.2.1 Rule of the magnetic flux



Example: the long current-carrying conductor

The total effect which the current in the conductor exerts upon the surrounding space is proportional to the electric current I and the conductor length l.

This total effect is distributed with growing distance r on a steadily increasing cylindrical external surface $A = 2 \cdot \pi \cdot r \cdot l$. This results in

influence density = magnetic field strength

$$\Leftarrow H = \frac{I \cdot l}{2 \cdot \pi \cdot r \cdot l} = \underline{\underline{\frac{I}{2 \cdot \pi \cdot r}}}$$

$$\underline{\underline{I = H \cdot 2 \cdot \pi \cdot r}}$$

$\Uparrow$

Special case of Ampere's law:

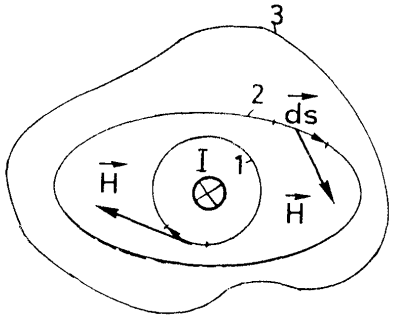
$$\boxed{I = \oint \vec{H} \cdot d\vec{s}}$$

Here, I is the current which punctuates the area framed by the integration path.

As a rule, it is recommended to integrate only along one field line; then $\vec{H}$ and $d\vec{s}$ always have the same direction and it is deemed:

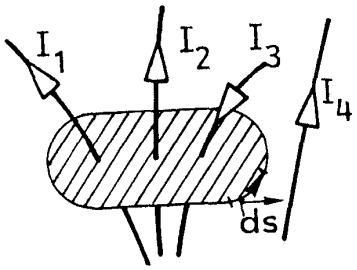
$$\boxed{I = \oint H \cdot ds}$$

The rule of the magnetic flux, discovered by the scientist Ampere, states that the result of this integration, when it takes place in a closed loop, is equal to the current which is enclosed by the integration path. It can be shown that the result of this integration is independent of which way is chosen. Therefore, it makes sense to choose the integration in such a way that the simplest possible mathematical expressions for the integration result.



Ampere's law does not only apply to an individual conductor. Through integration, a multitude of conductors having varying currents and/or varying current directions can be enclosed. In this case, the effective sum of these currents is described as current linkage (magnetomotive force). This term is frequently used in the subject of electrical machines.

$$\oint \vec{H} \cdot d\vec{s} = I_1 + I_2 - I_3 = \Theta$$



Ampere’s law can also be applied to massive conductors with unevenly distributed current density. In this case, however, it is necessary to insert the surface integral of the current density instead of the whole current.

$$I = \int_A \vec{J} \cdot d\vec{A}$$

$$\oint \vec{H} \cdot d\vec{s} = \int_A \vec{J} \cdot d\vec{A}$$

If there is no current present in the integration path, the result is 0.

$$\oint \vec{H} \cdot d\vec{s} = 0$$

The previously discussed individual cases can be expressed mathematically as follows:

1. In the case of a single conductor, it is deemed:

$$\oint \vec{H} \cdot d\vec{s} = I$$

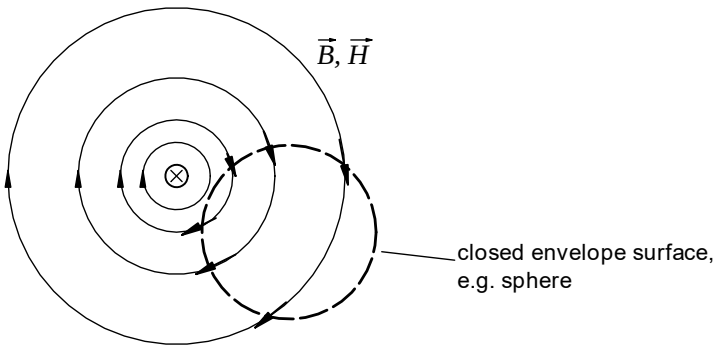
2. If several conductors are present, it is written:

$$\oint \vec{H} \cdot d\vec{s} = \sum_{k=1}^{n} I_k = \Theta$$

3. When examining the inner field of a conductor or if the current density is not distributed evenly, then it is deemed:

$$\oint \vec{H} \cdot d\vec{s} = \int_A \vec{J} \cdot d\vec{A} = \Theta$$

9.2.2 Magnetic vortex field



Magnetic field lines form a closed loop. They don't have a beginning or an end because there are no magnetic charges which (as in case of an electric field) could constitute the sources or sinks of the magnetic field.

The field lines of the magnetic induction, which enter in a volume through a self-contained, closed mantle, must exit at a different point.

$$\oint \vec{B} \cdot d\vec{A} = 0$$

This equation states that the magnetic field has no sources (solenoidality).

$$\oint \vec{H} \cdot d\vec{s} = \int_A \vec{J} \cdot d\vec{A}$$

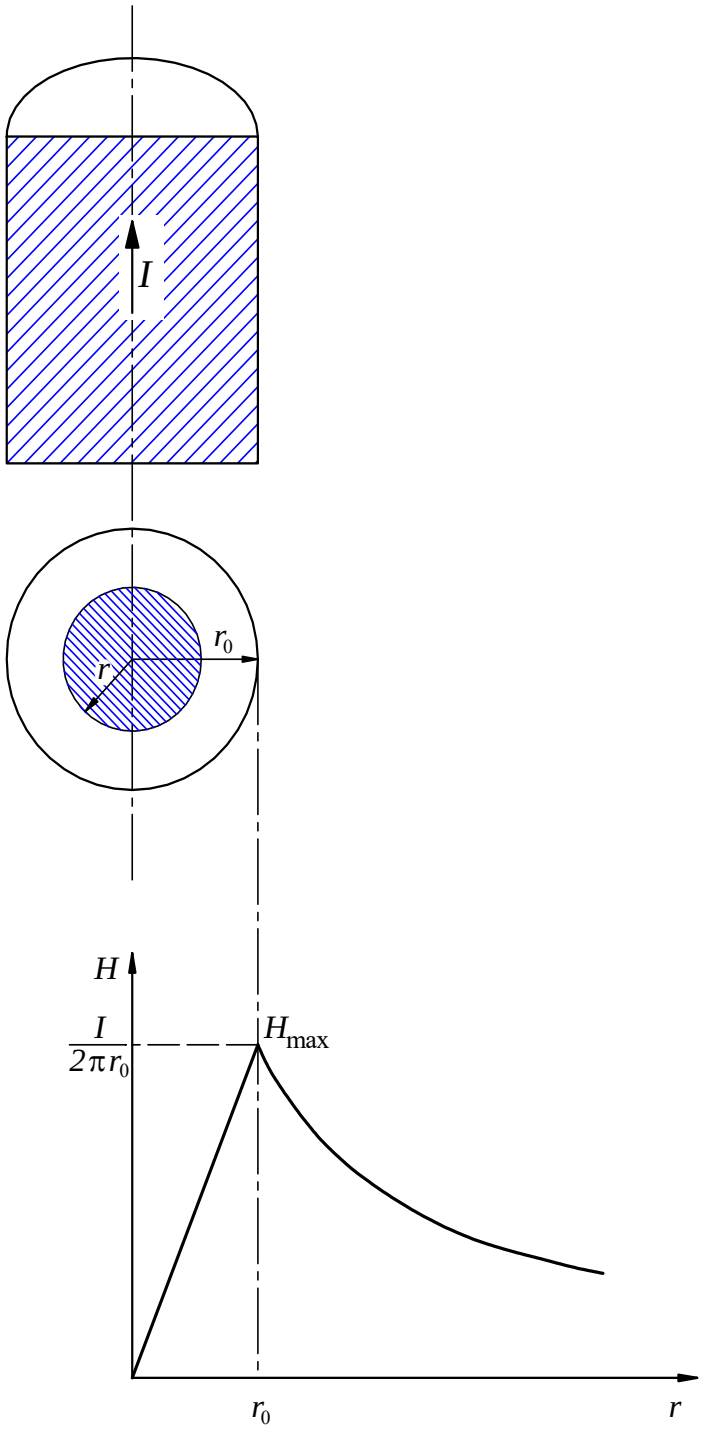
Ampere's law shows that the magnetic field has vortices.

The magnetic field is a source-free vortex field.

9.2.3 Magnetic field of a straight conductor

$r \geq r_0$ outside the conductor $H = \dfrac{I}{2 \cdot \pi \cdot r}$ (compare 9.1.6)

$r \leq r_0$ inside the conductor $\oint \vec{H} \cdot d\vec{s} = I'$



I' is the current which punctuates the area framed by the integration path.

I is the total current.

Due to the symmetry of the arrangement, H is the same at every point of the integration path (concentric circle).

$$H \cdot 2 \cdot \pi \cdot r = \frac{I}{\pi \cdot r_0^{\,2}} \cdot \pi \cdot r^2$$

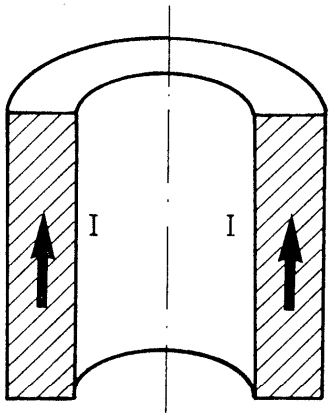
$$H = \frac{I \cdot r}{2 \cdot \pi \cdot r_0^{\,2}}$$

complete distribution

9.2.4 Magnetic field of an current-carrying tube

Another often-used structure is a tube carrying a current. In this case there is an area without current in the centre. We proceed on the assumption that the current density in the tube's wall remains constant. Therefore there are several areas to differentiate [4].

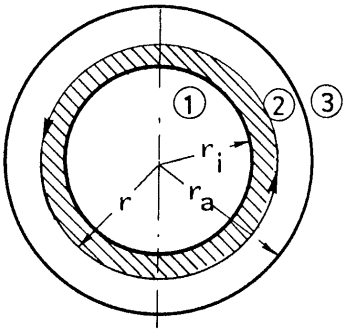
1. $0 \le r \le r_i$:



There is no current in the inner area.
Because no current is enclosed,
no field exists in this area.

$$\oint H_1 \cdot ds = 0 \Rightarrow \boxed{H_1 = 0}$$

2. $r_i \le r \le r_a$:



The area of the tube's wall is characterised by an increasing current. The closer to the outer wall, the greater the amount of enclosed current.

$$I_i = J \cdot A_i = \frac{I}{\pi \cdot \left(r_a^2 - r_i^2\right)} \cdot \pi \cdot \left(r^2 - r_i^2\right)$$

$$\oint H_2 \cdot ds = H_2 \cdot 2\pi \cdot r = I \cdot \frac{r^2 - r_i^2}{r_a^2 - r_i^2}$$

$$\Rightarrow H_2 = \frac{I}{2\pi \cdot r} \cdot \frac{\left(r^2 - r_i^2\right)}{\left(r_a^2 - r_i^2\right)}$$

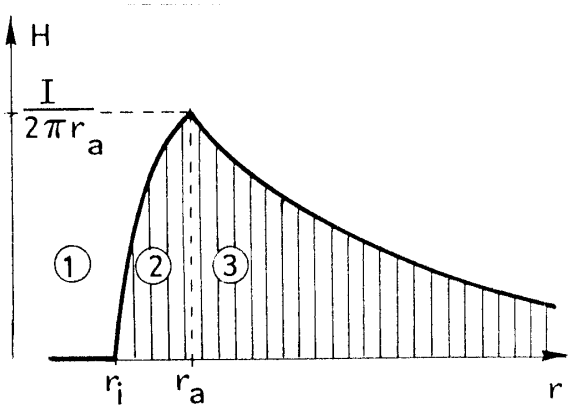
3. $r \ge r_a$:

As mentioned earlier, there is no increase of current outside the tube and it is deemed:

$$H_3 = \frac{I}{2\pi \cdot r}$$

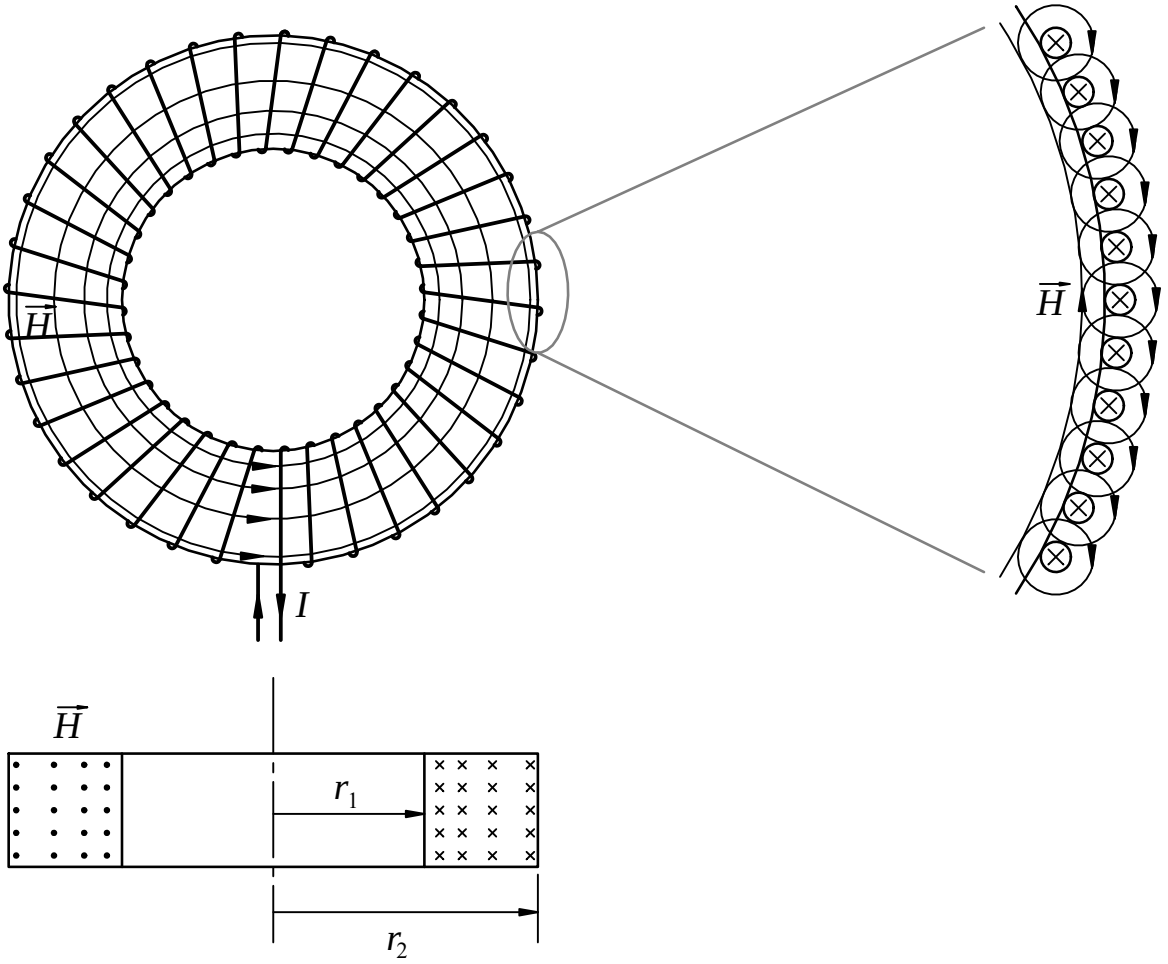
The field maximum is set on the outer surface of the tube:

$$H_{\max} = \frac{I}{2\pi \cdot r_a}$$



9.2.5 Magnetic field of the circular coil

N windings are very close to one another. The assembly is essen-tially cylindrically symmetrical. The resulting magnetic field ensues from the superposition of the magnetic fields of the individual windings.



$$\oint \vec{H} \cdot d\vec{s} = \sum I = \Theta \quad \Longrightarrow$$

$r<r_1$

$$H \cdot 2 \cdot \pi \cdot r = 0$$
$$\underline{\underline{H = 0}}$$

$r_1 \leq r \leq r_2$

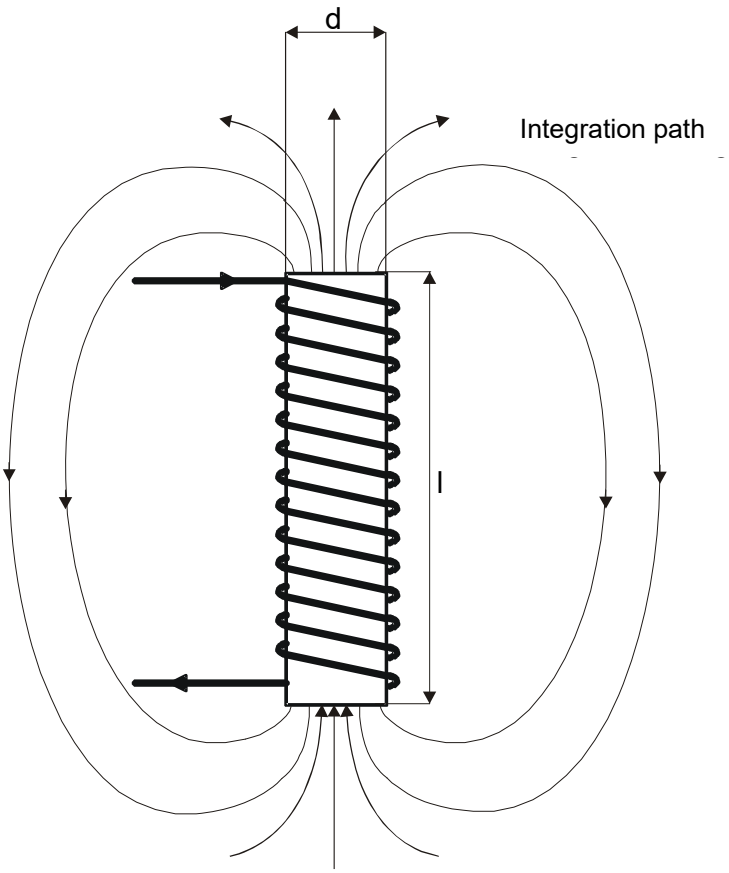
$$H \cdot 2 \cdot \pi \cdot r = N \cdot I$$
$$\underline{\underline{H = \frac{N \cdot I}{2 \cdot \pi \cdot r}}}$$

$r>r_2$

$$H \cdot 2 \cdot \pi \cdot r = N \cdot I - N \cdot I$$
$$\underline{\underline{H = 0}}$$

A magnetic field exists only inside the coil!

9.2.6 Magnetic field of the (long) cylindrical coil



l >> d

N windings are very close to one another. Field lines bundle on the inside of the coil all in the same way.

Therefore:
homogenous field with large field strength

Field lines close across the outer space. This space is infinitely large. ⇒

Low field line density ≅ low field strength

$$\oint \vec{H} \cdot d\vec{s} = H \cdot l + \int_{Exterior} \vec{H} \cdot d\vec{s} = \Theta \qquad \int_{Exterior} \vec{H} \cdot d\vec{s} << H \cdot l, \ \text{if} \ \ l >> d$$

$$H \cdot l \approx N \cdot I$$

$$\underline{\underline{H \approx \frac{N \cdot I}{l}}}$$

Comparison with the result for the circular coil:

A cylindrical coil curved into a circle with the length $l = 2 \cdot \pi \cdot r$

$$H = \frac{N \cdot I}{2 \cdot \pi \cdot r}$$

9.3 Biot-Savart law

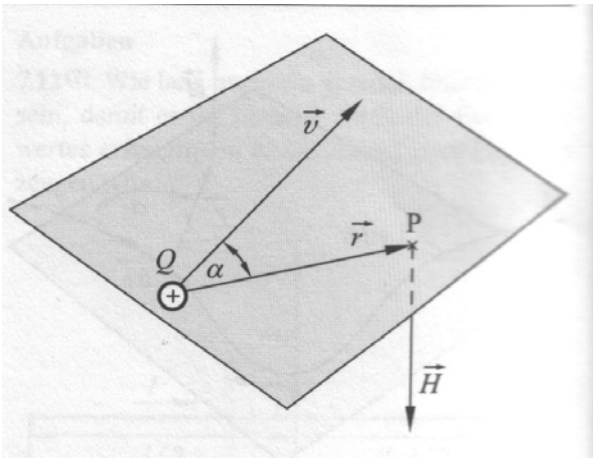
9.3.1 Magnetic field of moving charges

Ampere's law: can usually be analytically used only in simple symmetrical cases.

For other important technical cases: Biot-Savart law (can be derived from Ampere's law).

Approach: Every movement of an electric point charge Q causes a magnetic field strength H in the entire space [2]:

$$H = \left| \frac{Q \cdot v \cdot \sin\alpha}{4\pi \cdot r^2} \right|$$

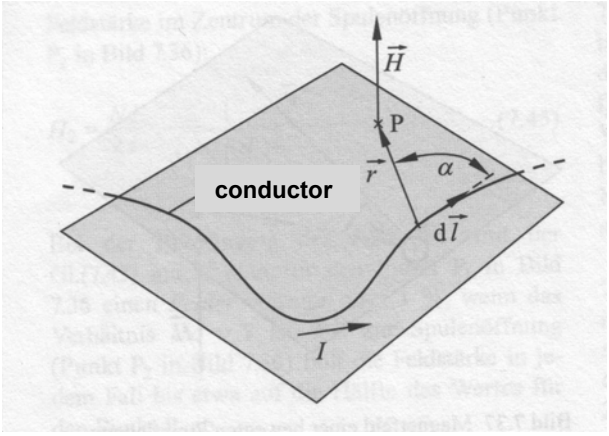


The field strength vector $\vec{H}$ is perpendicular to both $\vec{v}$ and $\vec{r}$. If the product $v \cdot r \cdot \sin\alpha$ is expressed as the vector product $\vec{v} \times \vec{r}$, the Biot-Savart law ensues:

$$\boxed{\vec{H} = \frac{Q}{4\pi \cdot r^3} \cdot \left(\vec{v} \times \vec{r}\right)}$$

Instead of a single point charge, we now want to observe an current-carrying conducter element with the length $d\vec{l}$. In it, the charge $dQ = I \cdot dt$ moves with the velocity $\vec{v} = d\vec{l}/dt$. At point P, the portion of the magnetic field strength ensues [2]:

$$d\vec{H} = \frac{I}{4\pi \cdot r^3} \cdot \left(d\vec{l} \times \vec{r}\right)$$

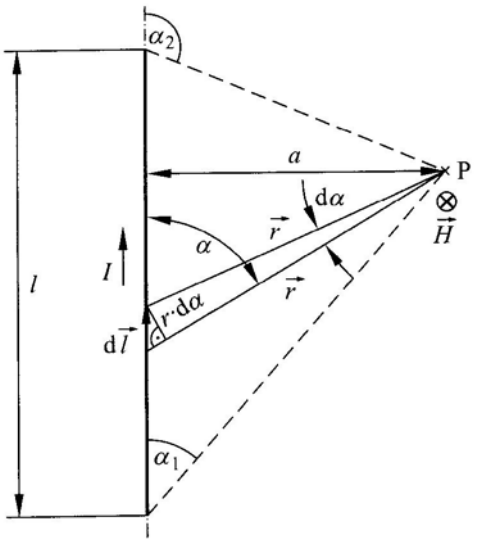


Examining important technical cases in which the point P and the conductor are located in one plane, all the field strength portions $d\vec{H}$ are perpendicular to this plane and the field strength $\vec{H}$ can be easily calculated by integrating the portions:

$$\vec{H} = \int_l d\vec{H} = \frac{I}{4\pi} \cdot \int_l \frac{d\vec{l} \times \vec{r}}{r^3}$$

Quantity: $H = \frac{I}{4\pi} \cdot \int_l \frac{\sin\alpha}{r^2} \cdot dl$

9.3.2 Magnetic field of a thin, straight conductor



The direction of the magnetic field strength results from the right-hand-rule.

The observed conductor has the minimal distance a from point P. If in the relation:

$$H = \frac{I}{4\pi} \cdot \int_l \frac{\sin\alpha}{r^2} \cdot dl$$

$\sin\alpha \cdot dl = r \cdot d\alpha$ and $r = \frac{a}{\sin\alpha}$ are replaced, it follows that the amount of the magnetic field strength is:

$$H = \frac{I}{4\pi \cdot a} \cdot \int_{\alpha_1}^{\alpha_2} \sin\alpha \cdot d\alpha$$

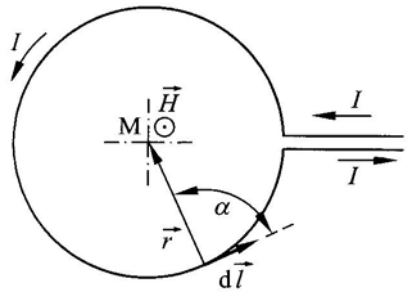
and therefore the solution is:

$$H = \frac{I}{4\pi \cdot a} \cdot (\cos\alpha_1 - \cos\alpha_2).$$

For the infinitely long conductor (α_1 = 0°, α_2 = 180°) we obtain: $H = \frac{I}{2\pi \cdot a}$.

This result was already derived from Ampere's law (compare 9.2.3) [2].

9.3.3 Magnetic field of a thin, circular conducting loop



When traversing the conductor loop the distance r to the middle point M remains constant. Furthermore, the angle α = 90° is also constant. Thus ensues [2]:

$$H = \frac{I}{4\pi} \cdot \int_l \frac{\sin\alpha}{r^2} \cdot dl = \frac{I}{4\pi \cdot r^2} \cdot \int_l dl = \frac{I}{4\pi \cdot r^2} \cdot 2\pi \cdot r$$

and it follows: $H = \frac{I}{2r}$

9.4 Material in a magnetic field / Magnetic characteristics

9.4.1 Physical characteristics

On the atomic scale, the electric field has an effect on material. At the same time the field itself is being altered by this material. This is inherently true for the magnetic field, as well. The magnetic behaviour of material is very complex and without further explanation, not conclusively describable. We will therefore restrict ourselves to a seminal description.

This description assumes that the magnetic effect of material is caused by ring currents. These ring currents can be traced to the movement of electrons in the atoms. Dipole momenta can be assigned to these ring currents. These dipole momenta are usually distributed with random orientation throughout the material. These dipole momenta have no effect outside of the material. If the material is brought into a preexisting magnetic field, an orientation of the dipole momenta occurs. The material 'reacts' to the magnetic field.

If there is no material in the space of the magnetic field, then the ratio is still relatively simple. The relationship between the magnetic flux density and the magnetic field strength H results from the equation:

$$\vec{B} = \mu_0 \cdot \vec{H}$$

μ_0 is the permeability of free space (magnetic constant).

If there is material present in the space of the magnetic field, an orientation of the dipole momenta occurs. This orientation of the dipole momenta is expressed by an additional magnetizing vector M:

$$\vec{B} = \mu_0 \cdot \left(\vec{H} + \vec{M}\right)$$

We speak of an additional excitation of the magnetic field in the material. This excitation is produced by the material alone and is only present in it. With the definition of the magnetization a third quantity exists to describe the magnetic field in material. To be able to continue working with the two quantities magnetic flux density B and magnetic field strength H it is necessary to define two more material-dependent quantities, relative permeability μ_r and susceptibility χ_m. Hereby the magnetization M is defined as:

$$M = \chi \cdot H$$

With the introduction of the relative permeability follows:

$$\vec{B} = \mu_0 \cdot \left(\vec{H} + \chi \cdot \vec{H}\right) = \mu_0 \cdot \vec{H} \cdot \left(1 + \chi\right) = \mu_0 \cdot \mu_r \cdot \vec{H}$$

The product of μ_0 and the relative permeability μ_r is also identified as absolute permeability (compare 9.1.7).

$$\vec{B} = \mu_0 \cdot \left(\vec{H} + \vec{M}\right) = \mu_0 \cdot \mu_r \cdot \vec{H} = \mu \cdot \vec{H}$$

9.4.2 Diamagnetic and paramagnetic materials

For most materials, it is deemed: $\mu_r \approx 1$

Diamagnetism

Examples: copper, silver, bismuth, lead, water, nitrogen, glass

$$\mu_r < 1;\ \mu_r = 1 + \chi;\ |\chi| << 1$$

If a diamagnetic material is inserted into a homogenous magnetic field which exhibits the magnetic flux density B_0 in a vacuum, a value $B < B_0$ is set inside the element. The reason for this is that the diamagnetic element at first does not exhibit any magnetic characteristics at all. This also applies on the atomic level. Here the electrons are arranged in pairs in such a manner that the magnetic effects, which are induced by the ring currents and the spin of the electrons, are completely nullified outwardly.

Through a change in the outer magnetic field, additional ring currents are induced in the atoms. This expresses itself through a larger or smaller circulation velocity of the electrons around the atomic nucleus. These ring currents are always oriented in such a way that they counteract their genesis, and therefore oppose the change in the outer magnetic field (Lenz's law - 10.1.1).

In a diamagnetic material which is brought into an outer magnetic field, the outer magnetic field and the one induced by ring currents are superimposed. Thus, the magnetic flux density in the diamagnetic element is reduced.

Paramagnetism

Examples: aluminium, silicon, platinum, tungsten, tantalum, oxygen

$$\mu_r > 1;\ \mu_r = 1 + \chi;\ \chi << 1$$

The atoms of paramagnetic materials have a magnetic dipole moment. In an arranged outer field they attempt to orient themselves in the direction of that field. This orientation, however, is not stable due to the atomic temperature gradient; the orientation of all atoms of a paramagnetic material at room temperature cannot be achieved even with very strong magnetic fields. Therefore only a weak magnetic effect is set outwardly.

diamagnetic materials		paramagnetic materials	
material	susceptibility χ	material	susceptibility χ
bismuth	$-157.000 \cdot 10^{-6}$	liquid oxygen	$3600.0 \cdot 10^{-6}$
lead	$-15.900 \cdot 10^{-6}$	platinum	$256.0 \cdot 10^{-6}$
copper	$-9.650 \cdot 10^{-6}$	tungsten	$67.5 \cdot 10^{-6}$
water	$-9.050 \cdot 10^{-6}$	aluminium	$24.0 \cdot 10^{-6}$
nitrogen	$-0.006 \cdot 10^{-6}$	tantalum	$18.2 \cdot 10^{-6}$
hydrogen	$-0.002 \cdot 10^{-6}$	oxygen	$1.8 \cdot 10^{-6}$

9.4.3 Ferromagnetic and ferrimagnetic materials

Ferromagnetism

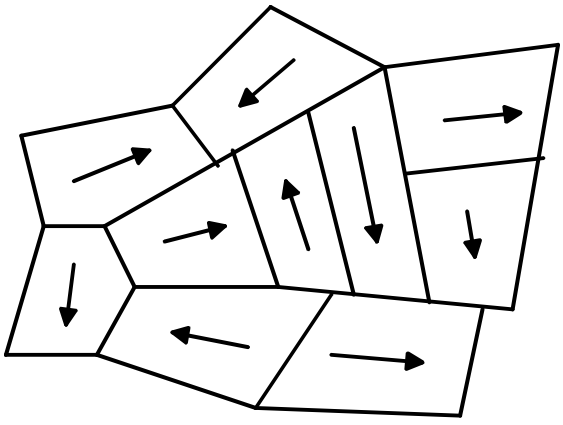
The molecules of these materials are magnetic dipoles, which orient themselves under the influence of the magnetic field and amplify it. Ferromagnetism occurs only in solid materials, not in liquids or gases, because this is not a characteristic of an atom, rather of the crystal structures of solid bodies.

Ferromagnetic materials: iron, cobalt, nickel

$$\mu_r = 10...10^5 \quad \text{(not constant)}$$

Ferromagnetism occurs when atoms affect each other (otherwise only paramagnetism would be present). Therefore in each case 10^6 … 10^9 elementary magnets in the so-called Weiss domains are oriented in the same direction; this results in an especially low energy content of the material. The elementary magnets within the Weiss domains orient themselves even without an outer magnetic field present.

Weiss domains



Weiss domains are areas, in which molecular dipoles orient themselves in a uniform direction. Collectively, these elementary magnets can compensate each other in their effect. However, they can predominantly have the same orientation, then a permanent magnet is present.

approximate volume : $10^{-1}...10^{-6}\,mm^3$
edge length : $10...500\,\mu m$

In a ferromagnetic material which is not magnetized, the orientation of the Weiss domains is absolutely random. If this material is brought into an external magnetic field, the border walls between the Weiss domains initially shift; those Weiss domains whose orientation coincidentally correlates with the outer magnetic field increase at the expense of their neighbouring domains. Initially, this process is reversible.

Upon further increase of the outer field's magnetic field strength *H*, irreversible shifts of the walls occur, whereby more and more border walls between individual Weiss domains disappear. The magnetic flux density *B* in the ferromagnetic material greatly increases. Ultimately, all Weiss domains turn toward the outer magnetic field; this condition is known as saturation.

If a ferromagnetic material is heated up to the so-called Curie temperature, the thermal energy leads to a molecule movement being powerful enough to break down the orientation of the Weiss domains. Hence, the material losses its magnetic characteristics.

Material		Curie temperature in °C
Iron	Fe	770
Cobalt	Co	1130
Nickel	Ni	358
Iron Oxide	Fe_2O_3	622

Antiferromagnetism

Although molecular magnet dipoles are present, they orient themselves anti-parallel due to their special crystal lattice structure and thereby nullify their outward effect.

Ferrimagnetism

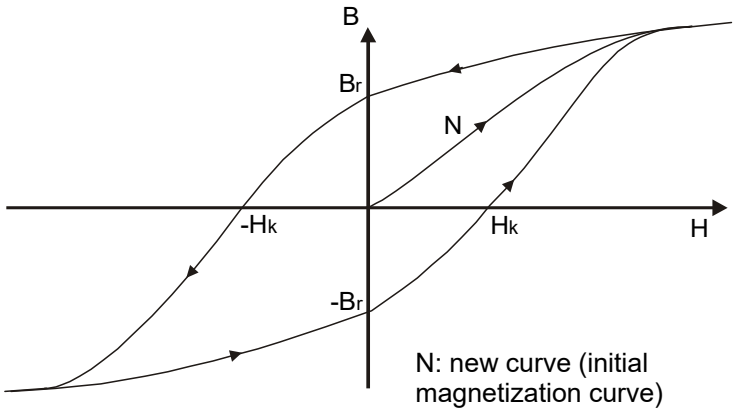
Anti-parallel oriented molecular magnet dipoles only partially compensate each other; therefore, one direction is predominant.

Ferrimagnetic materials: ferrite (Fe-, Mn-, Ni-, Zn-, Cu-, Mg-, Oxide sintered together at 1000 ... 1400 degrees)

$$\mu_r = 10^4 ... 10^6 \quad \text{(not constant)}$$

Ferrimagnetic materials behave essentially the same as ferromagnetic materials. However, the condition of saturation is reached at lower magnetic flux densities.

9.4.4 Magnetization curve



"Hysteresis" – curve

B-H- relationship

not linear
not definite

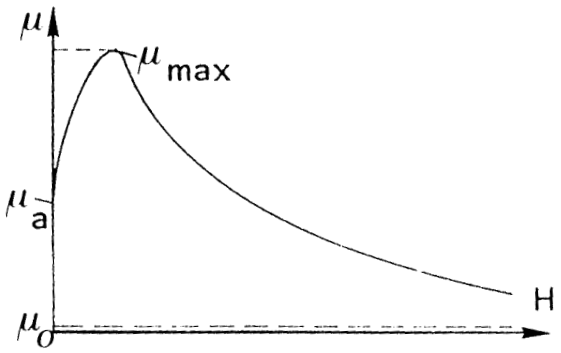
A piece of ferromagnetic material is inserted into a variable magnetic field. If the magnetic field strength is increased from zero, the magnetic flux density in the material initially follows the new curve (initial magnetization). After sharply increasing, the curve then levels out (saturation). If the magnetic field strength is further reduced, the magnetic flux density follows the upper curve. It intersects the axis of the magnetic flux density at the point B_R, this point is identified as remanence. Now the sign of the magnetic field strength is reversed. The curve of the flux density descends further. The point of intersection with the H - axis is identified as coercive field strength.

It is recognizable that in an alternating magnetic field, the two outer branches of the so-called hysteresis curve are passed through. This curve encompasses an area which is proportional to the hysteresis losses which occur during the process (10.3.4).

Absolute permeability is the quotient B over H. In the case of ferromagnetic material relative and absolute permeability are no longer constants.

$$\mu = \mu(H) \neq const$$

The adjacent diagram shows a typical progression of the absolute permeability for a soft magnetic material [2].



In summary:

The elementary magnets orient themselves with increasing field strength.
When all magnets are oriented saturation $dB/dH = \mu_0$

When switching off the current (more exact: reducing the field strength *H* to zero):
Remanence – induction B_r (remanere = remain)

To make the induction disappear, a magnetic field strength with opposite polarization - H_k is needed.
coercive – field strength H_k (coercere = coerce)

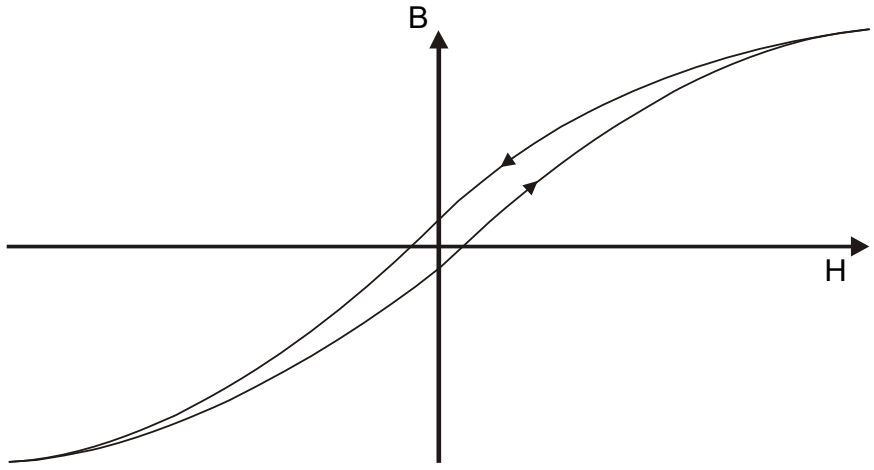
The induction follows the magnetic field strength with a time delay:
Hysteresis

Hard magnetic materials

broad hysteresis - loop, large B_r, large H_k
for all applications without change of direction, for example permanent magnets, ferrites

Soft magnetic materials

narrow hysteresis - loop, small B_r, small H_k, therefore the magnetization is easily reversible



For all applications with magnetic alternating fields (electrical machines, transformers), e.g. iron.

With soft magnetic materials one calculates often approximately with a definite B- H-relation

9.5 Magnetic circuit

9.5.1 Examples of magnetic circuits

Many power generating systems are based on utilization of the magnetic field. For the calculation of the mechanical quantities force and torque, a calculation of the quantities of the magnetic field is required. This can be done with computer programs, which can use the finite element method to calculate the magnetic flux density or the magnetic field strength. These calculation methods shall not be further discussed during this lecture.

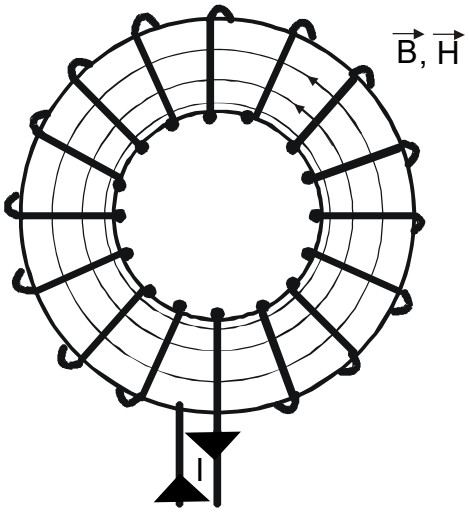
Another possibility is using neglections, which will lead to a relatively simple analytical calculation of a magnetic circuit. With this method, one can use the calculation procedures or methods which we already know from electric circuits. Current linkages (magnetic “voltages”) and magnetic resistances are defined. The magnetic flux corresponds to the current in an electric circuit.

Here, a fundamental difference between an electrical circuit and a magnetic circuit shall be pointed out. For an electrical circuit, the assumption can be made that the current is carried only in a conductor due to the high conductance difference between non-conductor and conductor. The difference between the conductance of ferromagnetic materials and air, however, is minimal. Parts of the magnetic flux will close over the air. This is identified as leakage flux.

Magnetic circuits can feature nodes and meshes (loops) like electrical circuits. A simple transformer can, for example, be expressed by a magnetic circuit with one loop. An electrical machine will have many junctions and loops. Another distinguishing characteristic is the air gap. A transformer is realized without an air gap. An electrical machine must have an air gap to separate rotor and stator.

9.5.2 Current linkage and magnetic voltage

Example: circular loop



Ampere's law:

$$\oint \vec{H} \cdot d\vec{s} = N \cdot I = \Theta$$

$$\Theta = Current\ linkage$$

When one integrates only along a part of a magnetic circuit, when therefore the integral is not a closed loop integral anymore, then the result is identified as:

V = magnetic voltage

$$\int_{P_1}^{P_2} \vec{H} \cdot d\vec{s} = V_{12} \qquad V = \text{magnetic voltage}$$

$$\text{Unit}\left[\Theta\right] = \text{Unit}\left[V\right] = A$$

9.5.3 Magnetic flux

In many assemblies (e.g. circular loop) the magnetic field is guided in a rather precisely limited area. All induction lines pass through this area. Obvious: definition of another integral quantity:

$$\boxed{\Phi = \int_A \vec{B} \cdot d\vec{A}}$$

$$\Phi = \text{magnetic flux}$$

B is a measure of the field line density
Φ is a measure of the number of field lines

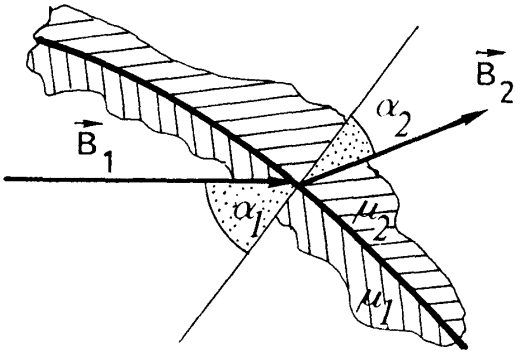
2 Names for $\vec{B}$: $\vec{B}$ = Induction
$\vec{B}$ = magnetic flux density

$$\text{Unit}\left[\Phi\right] = T \cdot m^2 = \frac{Vs}{m^2} \cdot m^2$$

$$\text{Unit}\left[\Phi\right] = Vs = Wb\,(\text{Weber})$$

9.5.4 Magnetic field at boundaries

In the electrical field we have ascertained that on the boundary between two dielectrics with differing permittivities ε the electrical field strength lines are broken. Similarly, we now examine the field lines of the magnetic flux density B and the magnetic field strength H on the boundary between two materials with different permeability [4, 5].



The vector of the magnetic induction B_1 in the medium 1 at a point on the boundary generates the angle α_1 with the normals at this point on the boundary. Amount and direction of the vector B_2 in the medium 2 at the same point, which means on the other side of the boundary, are accepted differently from B_1. The two vectors are disassembled into a tangential component and a normal component. To describe the relationship between the tangential and the normal components of both sides, Ampere's law (9.2.1) and the fact, that the magnetic field has no sources (solenoidality: 9.2.2), are available.

With the solenoidality it follows that the normal components of the magnetic flux density are equal on both sides:

$$B_{n1} = B_{n_2}$$

With Ampere's law it can be derived that the tangential components of the magnetic field strength must also be equal:

$$\oint \vec{H} \cdot d\vec{s} = \theta \qquad \rightarrow \qquad H_{t1} = H_{t_2}$$

Similar to the electrical field, the refraction angle can be calculated with the absolute permeability on both sides:

$$\tan\alpha_1 = \frac{B_{t_1}}{B_{n_1}} = \frac{\mu_1 \cdot H_{t_1}}{B_{n_1}} \qquad \tan\alpha_2 = \frac{B_{t_2}}{B_{n_2}} = \frac{\mu_2 \cdot H_{t_2}}{B_{n_2}}$$

$$\frac{\tan\alpha_1}{\tan\alpha_2} = \frac{\mu_1 \cdot \cancel{H_{t_1}}}{\cancel{B_{n_1}}} \cdot \frac{\cancel{B_{n_2}}}{\mu_2 \cdot \cancel{H_{t_2}}} \qquad \Rightarrow \qquad \boxed{\frac{\tan\alpha_1}{\tan\alpha_2} = \frac{\mu_1}{\mu_2}}$$

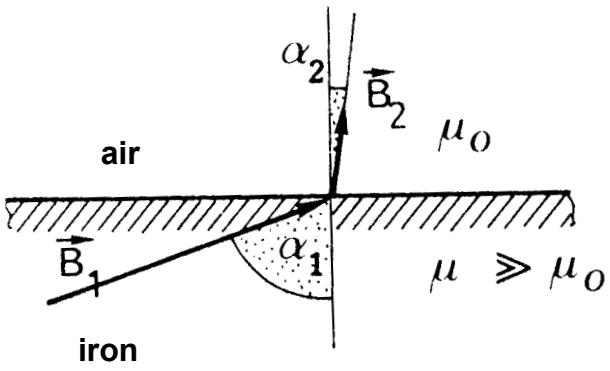
The crossing of the magnetic field from air to steel shall be observed as a frequent application. This crossing is characterized by a strong change in absolute permeability. The absolute permeability is equal to the natural constant μ_0 in air. In iron, the absolute permeability is 1000 to 5000 times higher. In comparison to the permeability of air it is almost infinite. From this, it can be derived that the exit angle of the field lines on air is about 90 °, as shown below [4, 5].

$$\tan\alpha_2 = \tan\alpha_1 \cdot \frac{\mu_2}{\mu_1} \approx 0$$

$$\alpha_2 \approx 0, \text{ because } \frac{\mu_2}{\mu_1} \approx 0$$

$$H_{t_1} = H_{t_2} \Rightarrow \frac{B_{t_1}}{\mu_1} = \frac{B_{t_2}}{\mu_2}$$

$$\Rightarrow B_{t_1} = \frac{\mu_1}{\mu_2} \cdot B_{t_2} \rangle\rangle B_{t_2}$$



9.5.5 Magnetic resistance and conductance of the magnetic circuit

The integral quantities current linkage Θ and magnetic flux Φ are primarily used when the magnetic field is conducted by materials with high permeability (iron).

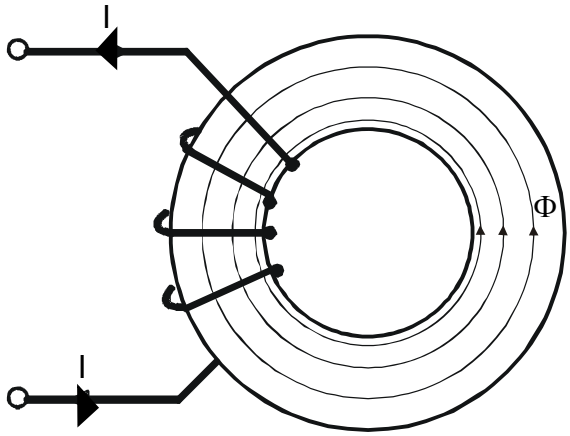
Example: coil (loop) with closed iron core

Calculation with medium values for H and B
l = medium iron path length

$$\Theta = \oint \vec{H} \cdot d\vec{s} \quad \rightarrow \Theta = H \cdot l \rightarrow H = \frac{\Theta}{l}$$

$$\Phi = \int \vec{B} \cdot d\vec{A} \rightarrow \Phi = B \cdot A \text{ with } B = \mu \cdot H$$

$$\Phi = \frac{\mu \cdot A}{l} \cdot \Theta$$



Comparison: voltage U_0 is applied to an electrical conductor with the length l, the conductivity κ, and the cross section A $\rightarrow I = \frac{\kappa \cdot A}{l} \cdot U_0$

Therefore, it is analogically defined:

$$G = \frac{\kappa \cdot A}{l} \qquad \Lambda = \frac{\mu \cdot A}{l} \quad \text{magnetic conductance}$$

$$R = \frac{l}{\kappa \cdot A} \qquad R_m = \frac{l}{\mu \cdot A} \quad \text{magnetic resistance}$$

$$U_0 = R \cdot I \qquad \boxed{\Theta = R_m \cdot \Phi} \quad \text{“ Ohm's law of the magnetic circuit“}$$

$$\text{Unit}\left[R_m\right] = \frac{A}{Vs}$$

Kirchhoff's current law: $\sum I_\nu = 0 \qquad \sum \Phi_\nu = 0$

Kirchhoff's voltage law: $\sum U_\nu = 0$

$$\oint \vec{H} \cdot d\vec{s} = \sum V_\nu = \Theta$$

$$\sum V_\nu - \Theta = 0$$

The formal difference with Kirchhoff's voltage law is practically without consequence if the current linkage Θ in magnetic cases is equally considered as an impressed voltage U_0 in the electric cases.

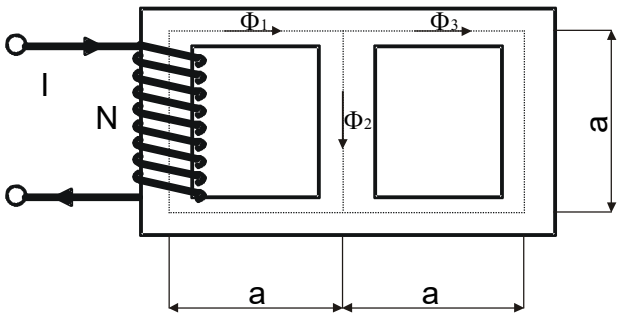
Analogies:

U_0	↔	Θ
U	↔	V
I	↔	Φ
R	↔	R_m
κ	↔	μ

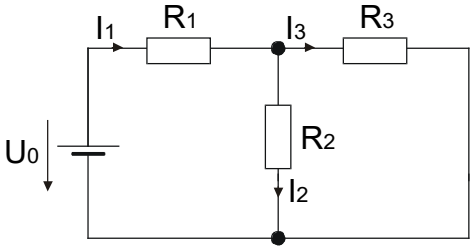
With these analogies the voltage divider rule and the current divider rule are applicable to magnetic circuits.

Example:

iron cross section is uniform



electric analogue



$$\Phi_1 = \frac{\Theta}{R_{m1} + \frac{R_{m2} \cdot R_{m3}}{R_{m2} + R_{m3}}} \qquad I_1 = \frac{U_0}{R_1 + \frac{R_2 \cdot R_3}{R_2 + R_3}}$$

$$\Theta = N \cdot I; R_{m1} = \frac{3a}{\mu \cdot A}; R_{m2} = \frac{a}{\mu \cdot A}; R_{m3} = \frac{3a}{\mu \cdot A}$$

$$\Phi_1 = \frac{\mu \cdot A \cdot N \cdot I}{3a + \frac{a \cdot 3a}{a + 3a}} = \frac{\mu \cdot A \cdot N \cdot I}{3{,}75a} = \Phi_1$$

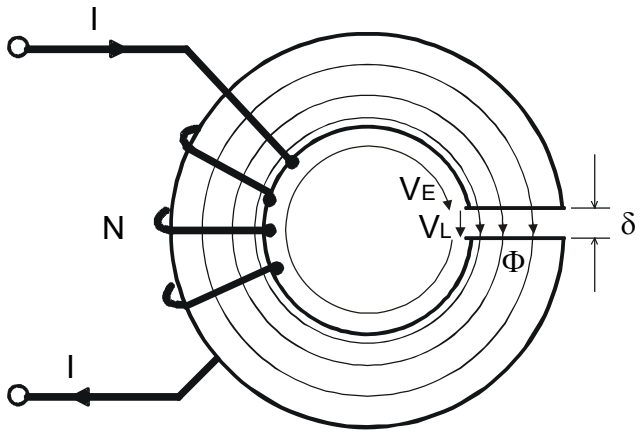
$$\frac{\Phi_2}{\Phi_1} = \frac{\frac{1}{R_{m2}}}{\frac{1}{R_{m2}} + \frac{1}{R_{m3}}} \qquad \frac{I_2}{I_1} = \frac{\frac{1}{R_2}}{\frac{1}{R_2} + \frac{1}{R_3}}$$

$$\Phi_2 = \frac{\frac{\mu \cdot A}{a}}{\frac{\mu \cdot A}{a} + \frac{\mu \cdot A}{3a}} \cdot \frac{\mu \cdot A \cdot N \cdot I}{3{,}75a} = \frac{\mu \cdot A \cdot N \cdot I}{5a} = \Phi_2$$

$$\Phi_3 = \Phi_1 - \Phi_2 \qquad I_3 = I_1 - I_2$$

$$\Phi_3 = \left(\frac{1}{3{,}75} - \frac{1}{5} \right) \cdot \frac{\mu \cdot A \cdot N \cdot I}{a} = \frac{\mu \cdot A \cdot N \cdot I}{15a} = \Phi_3$$

9.5.6 Magnetic circuit with air gap



In case of moving magnetic circuits (e.g electrical machines), the iron core is usually interrupted by an air gap.

here: air gap length δ

$\delta << l$

l = iron path length ≈ total length

$$\oint \vec{H} \cdot d\vec{s} = \sum V_\nu = \Theta$$

$$\left(R_{mE} + R_{mL}\right) \cdot \Phi = \Theta$$

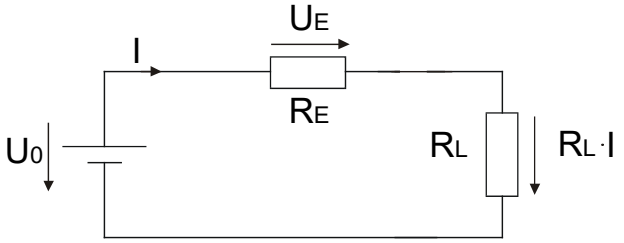
$$\left(\frac{l}{\mu_r \cdot \mu_0 \cdot A} + \frac{\delta}{\mu_0 \cdot A}\right) \cdot \Phi = \Theta$$

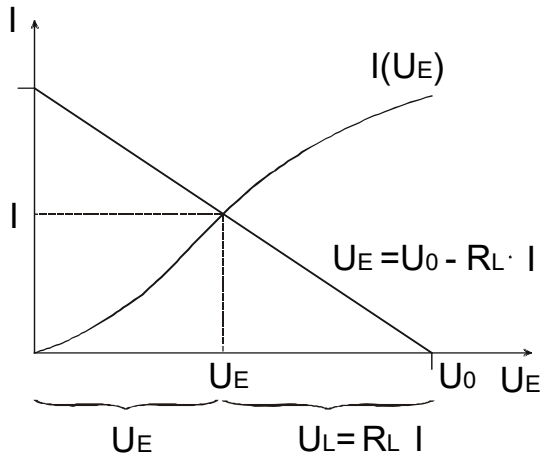
$\frac{l}{\mu_r} \approx \delta$ (of the same order of magnitude)

μ_r not constant

$$\left(\frac{l}{\mu_r} + \delta\right) \cdot \frac{\Phi}{\mu_0 \cdot A} = \Theta$$

Electric analogue:

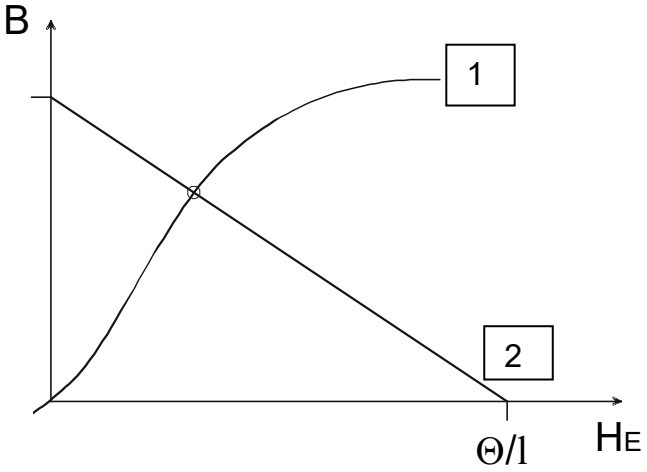




Resistor R_E is not linear. Solution by graphically determining the bias point

For the magnetic circuit this method must be slightly modified because not the relation $\Phi(V_E)$,but the magnetization curve $B(H_E)$ is specified for magnetic materials.

Example:
magnetization curve for soft magnetic material, hysteresis neglected



Determination of the bias point:

$$\oint \vec{H} \cdot d\vec{s} = \Theta$$

$$H_E \cdot l + H_L \cdot \delta = \Theta$$

$$H_E \cdot l + \frac{B}{\mu_0} \cdot \delta = \Theta$$

$$\underline{\underline{H_E = \frac{\Theta}{l} - \frac{\delta}{\mu_0 \cdot l} \cdot B}}$$

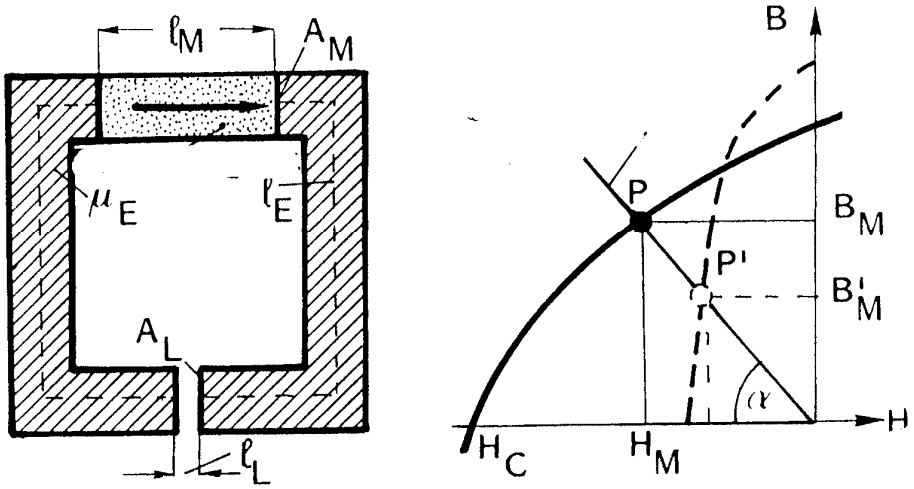
1. This relationship between *B* and H_E describes the magnetic characteristics of the magnetic material.
2. This relationship between *B* and H_E results from the current linkage Θ and the geometric dimensions.

Bias point ⟶ point of intersection

9.5.7 Magnetic circuits with permanent magnets

A typical magnetic circuit with a permanent magnet consists of the permanent magnet itself, a ferromagnetic material to conduct the magnetic flux, and an air gap in which the majority of the magnetic energy is stored. The magnet has the height l_m, the length of the iron path is l_E, and the length of the air gap is l_L. The cross-section area is constant and equals A_M in the whole magnetic circuit .

In connection with the characteristic curve of the magnetic material, the bias point P of the magnetic circuit is thereby completely defined. How the bias point is calculated will be shown in this section.



It is important that the orientations of the magnetic field strength H and the magnetic flux density B in the magnetic circuit are the same in the iron and the air gap. In the magnet, however, these two quantities have opposing orientations.

Ampere's law now reads as follows:

$$\oint \vec{H} \cdot d\vec{s} = \theta$$

Thereby a relationship between the magnetic field strength of a magnetic circuit and the excited current linkage is defined. However, in this given magnetic circuit no current is present. The application of Ampere's law must therefore result in the value 0. Due to the permanent magnet a magnetic field strength is present inside and outside of the magnet. Thus it must be deemed:

$$\oint \vec{H} \cdot d\vec{s} = \int \vec{H}_a \cdot d\vec{s} + \int \vec{H}_i \cdot d\vec{s} = 0 \Rightarrow \int \vec{H}_a \cdot d\vec{s} = -\int \vec{H}_i \cdot d\vec{s}$$

It can be recognized that the magnetic field strength in a magnetic circuit with a permanent magnet experiences a sign change. With a positive orientation of the magnetic flux density B in the magnet, the magnetic field strength H is oriented in the negative direction.

Now to the calculation of the magnetic circuit. The total decrease of magnetic voltage in the magnetic circuit results due to Ampere's law as follows:

$$H_M \cdot l_M + H_E \cdot l_E + H_L \cdot l_L = 0$$

To further simplify, we assume that, based on the relatively large permeability of iron, the field strength in the iron can be practically neglected (this approximation is very often valid):

$$H_M \cdot l_M + H_L \cdot l_L = 0$$

The continuity of the flux is expressed with the following equation. The effect of dispersion, which in magnetic circuits with permanent magnets can frequently not be neglected, is considered by the additional factor k_s. This factor is always <1, which means that only a portion of the flux generated by a permanent magnet is available in the air gap:

$$B_L \cdot A_L = k_s \cdot B_M \cdot A_M$$

The combination of the two equations results in the so-called operation line. It is recognizable that it is defined only by the geometrical data of the circuit:

$$H_M = -H_L \cdot \frac{l_L}{l_M} = -k_s \cdot \frac{l_L}{l_M} \cdot \frac{A_M}{A_L} \cdot \frac{B_M}{\mu_0} \text{ or: } B_M = -\frac{\mu_0}{k_s} \cdot \frac{l_M}{l_L} \cdot \frac{A_L}{A_M} \cdot H_M$$

The gradient angle of this operation line can be calculated as follows:

$$\tan\alpha = -\frac{\mu_0}{k_s} \cdot \frac{l_M}{l_L} \cdot \frac{A_L}{A_M}$$

To define the exact bias point of the magnetic circuit, the characteristic curve of the material applied has to be available. For materials which are often used today (rare earth elements) the characteristic curve of the magnet is linear. The calculation of the intersection point of the operation line and the characteristic curve of the magnet is very easily possible in this case.

Aim of the optimization of a magnetic circuit with a permanent magnet is to realize a maximum energy density in an air gap with given dimensions with a permanent magnet with minimum volume. The magnetic flux density is given by:

$$B_L = k_s \cdot \frac{A_M}{A_L} \cdot B_M$$

$$B_L = \mu_0 \cdot \frac{l_M}{l_L} \cdot (-H_M)$$

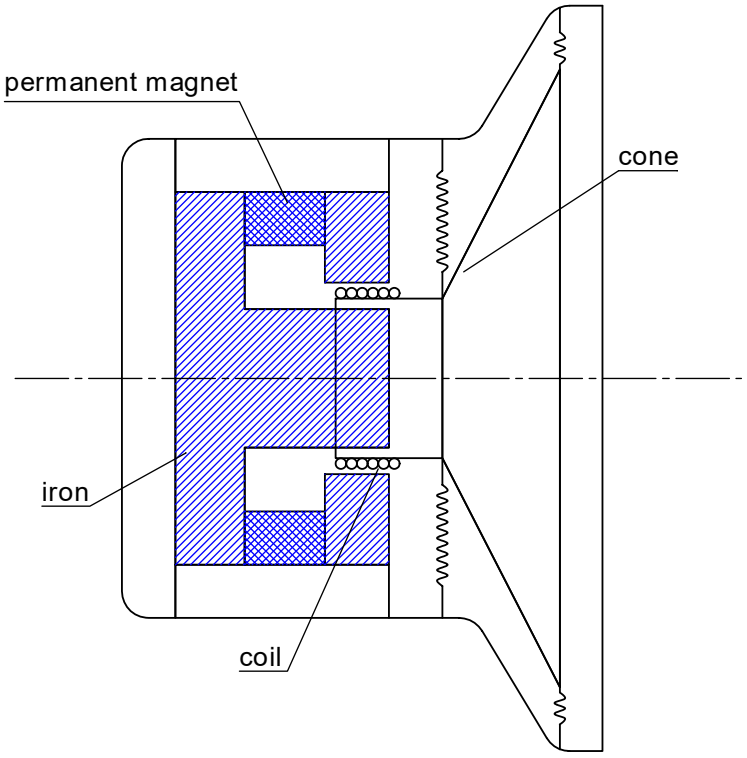
Multiplying both equations leads to:

$$B_L{}^2 = \mu_0 \cdot k_s \cdot \frac{l_M}{l_L} \cdot \frac{A_M}{A_L} \cdot B_M \cdot (-H_M) = \mu_0 \cdot k_s \cdot \frac{V_M}{V_L} \cdot B_M \cdot (-H_M)$$

This relationship shows the influence of both the volume of the magnet V_M and the volume of the air gap V_L and the influence of the magnet's energy density on the flux density in the air gap. Looking at the units one can recognize, that the product $B \cdot H$ is an energy density (10.3.2):

$$[B]\,[H] = \frac{Vs}{m^2} \cdot \frac{A}{m} = \frac{Ws}{m^3}$$

Examples for applications of magnetic circuits with permanent magnets are electrical machines, moving coil instruments, and loud speakers.



10. Time-variable magnetic fields

10.1 Law of induction, Faraday's law

10.1.1 Induction process and Lenz's law

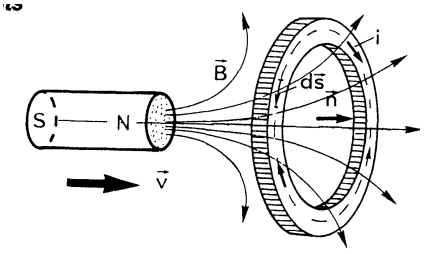
The experiments conducted to explore the induction process are owed to the scientist Faraday. He conducted experiments in which he moved a permanent magnet with a north and a south pole relative to a circular electrical conductor.

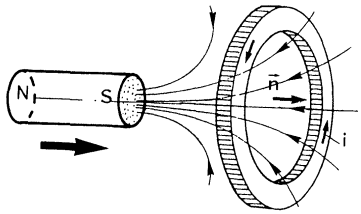
The essential result of these experiments was that through the movement of the magnet, a flux change occurred in the circular conductor, which in turn caused induction of currents in the circular conductor. In essence, the induction process is not dependent on the flux intensity, but on the flux change.

The definition of flux is well known:

$$\Phi = \int_A \vec{B} \cdot d\vec{A}$$

In these experiments, the north or the south pole can now be oriented in the direction of the ring. It is determined that the flux is positive when the area vector n and the vector of the magnetic flux density B are oriented in same direction. In the other case the flux is negative [4].





$$\frac{d\Phi}{dt} > 0 \qquad \frac{d\Phi}{dt} < 0$$

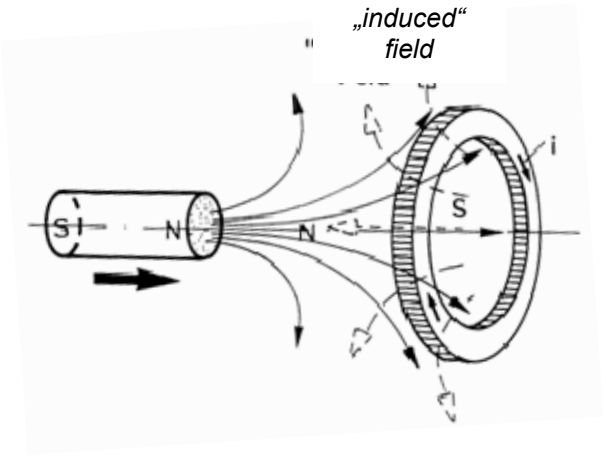
If the magnet is moved in the direction of the ring as shown in the left picture then the flux increases obviously, the flux change is therefore positive. The flux itself, as well. If the magnet is moved away from the ring, the flux itself remains positive; the change, however, is negative because the flux decreases.

In the right picture, the south pole of the magnet is facing the ring. The flux itself is negative. If the magnet is moved toward the ring, the flux change is negative, if the magnet is moved away from the ring, the flux change is positive.

Depending on the orientation of the magnetic field and the direction of motion, a voltage, and thereby a current, is induced in the ring. The current direction must therefore be different in both pictures. The question about the current direction is answered by the so-called Lenz's law.

A fundamental principle of physics is: actio equals reactio. This principle inevitably ensues from the law of conservation of energy. If this basic principle is applied to the process of electromagnetic induction, the following can be considered [4]:

The direction of the induced current will always be such that this current excites a magnetic field, which works against the external change. If, for example, the flux change is positive as in the adjacent picture, the induced field of the current will be opposite of the permanent magnet field. If the current would flow in the other direction, the outer field would be amplified by the field of the current, which means the flux change would increase. This in turn would lead to a larger induced current, etc...



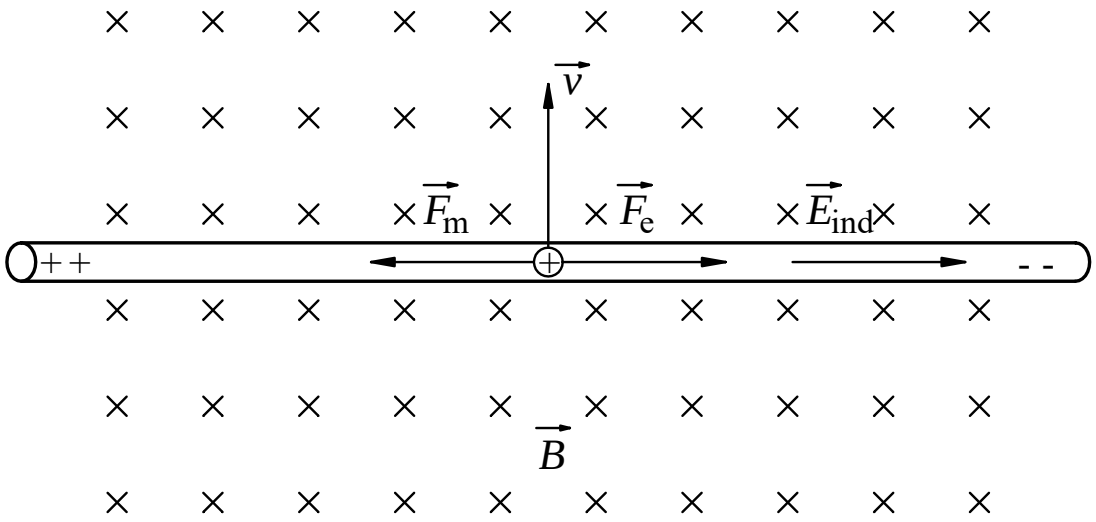
The energy in the system would increase itself after an initial excitation without external influence. And this is not possible according to the law of conservation of energy.

10.1.2 Calculation of the induced voltage for a moving conductor

We have already discussed the force effect on the electrified conductor (9.1.3). For a better understanding of the induction processes, we will calculate the force effects on individual charges. An external magnetic field shall be given; we won't have to consider its source at this point. This field is constant. In this field, a charge moves with the velocity v.

The force on the charge, the so-called Lorentz force F_m, results from the vector product of the velocity v and the magnetic field B multiplied by the charge amount Q, which is moved in the field.

We apply the Lorentz force to all charges which are located in a moving conductor. This conductor moves at a velocity v perpendicular to a constant magnetic field. To further simplify, we assume that the magnetic field in the observed space should be homogenous.



As shown in the picture, a charge separation occurs in the conductor. Induced by the Lorentz force, the negative charges, the free electrons of the conductor, are moved to the right end of the conductor. Thus ensues a positive charge on the left side of the conductor.

The Lorentz force acts upon the charges $\vec{F}_m = Q \cdot \vec{v} \times \vec{B}$.

This leads to the described shift of electrons to the right side of the conductor. Based on the two charges now present, a Coulomb force is created between these two charge areas, which counteracts the Lorentz force: $\vec{F}_e = Q \cdot \vec{E}_{ind}$

In a stationary condition, these two forces will neutralize one another

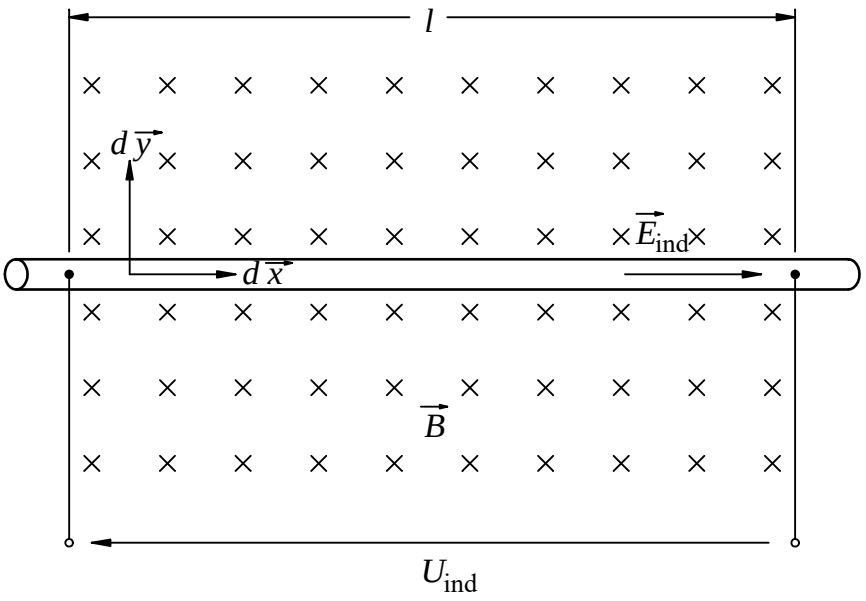
$$\vec{F}_e + \vec{F}_m = 0$$
$$Q \cdot \vec{E}_{ind} + Q \cdot \left(\vec{v} \times \vec{B}\right) = 0$$

It can be recognized in this equation that there is an equivalent to the electric field strength $\vec{E}_{ind}$, which is defined by the cross product of the velocity and the magnetic flux density. This electric field strength is identified as induced electric field strength:

$$\vec{E}_{ind} = -\vec{v} \times \vec{B}$$

We calculate the corresponding electric voltage using the line integral:

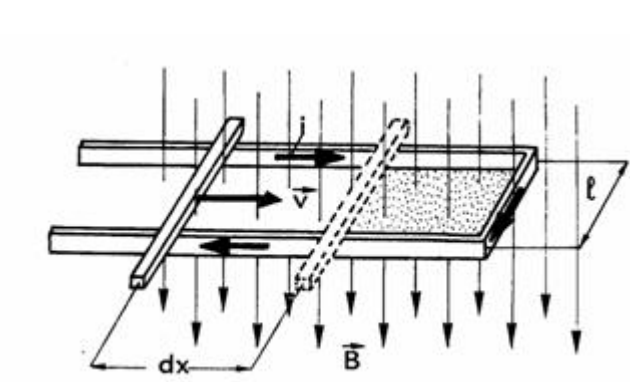
$$u_{ind} = \int_0^P \vec{E}_{ind} \cdot d\vec{x} = -\int_0^P \left(\vec{v} \times \vec{B}\right) \cdot d\vec{x}$$



If the two vectors velocity and magnetic flux density are perpendicular to each other then the calculation of the induced voltage is simplified (whereby l is the effective length of the conductor):

$$u_{ind} = -v \cdot B \cdot l$$

With this derivation of the induced voltage, the term magnetic flux has not yet been necessary. Only the velocity of the conductor, its length, and the magnetic flux density B have been required. If this conductor is considered to be part of a closed circuit, then another method for calculating the induced voltage can be derived.



In this picture, the conductor is a connecting element in a slide bar assembly The conductor can be moved freely. It moves with the velocity v. An area is defined by this conductor loop on which the definition of the magnetic flux can now be applied [4].

The velocity of the conductor and its length together result in the surface change of this conductor loop. Thus it is deemed:

$$u_{ind} = -v \cdot B \cdot l = -\frac{dx}{dt} \cdot B \cdot l = -B \cdot \underbrace{l \cdot \frac{dx}{dt}}_{\frac{dA}{dt}} = -B \cdot \frac{dA}{dt}$$

With the definition of the magnetic flux follows:

$$\boxed{u_{ind} = -\frac{d\Phi}{dt}}$$

Law of induction (Faraday's law)

The direction of the arrow of the induced voltage is clockwise toward the arrow of the field (see 10.1.1)

> The voltage induced in a conducting loop is equal to the negative rate of change of the magnetic flux which passes through it.

The law of induction applies universally, irrespective of the causes of the magnetic flux change.

The induced electricity is oriented in such a way that it counteracts its cause, namely the decrease of magnetic flux.

In general:

> Induced voltages and currents have such an orientation (direction) that they counteract the cause of their origin (Lenz's law)

10.1.3 Stationary conductor in the time-variable magnetic field

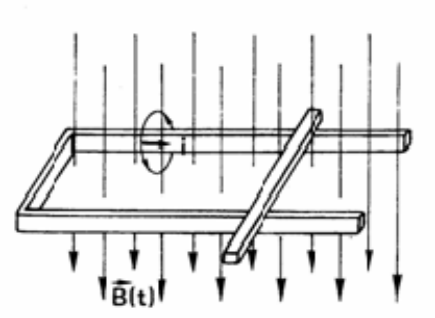
In the example above (10.1.2), we assumed the magnetic field as time constant. The change in the magnetic flux was caused by the change in the area of the conductor loop. We will deal with a flux change in this section, which is caused by a change in the magnetic field. This example is equivalent to the conditions present in a transformer, for example.

The location of the conductor loop is constant; the external magnetic field is a function of time. The flux in the conductor loop is therefore calculated with the formula:

$$\Phi(t) = B(t) \cdot A$$

In this case, the induced voltage is directly proportional to the rate of change in the magnetic flux density [4]:

$$u_{ind} = -\frac{d\Phi}{dt} = -A \cdot \frac{dB}{dt}$$



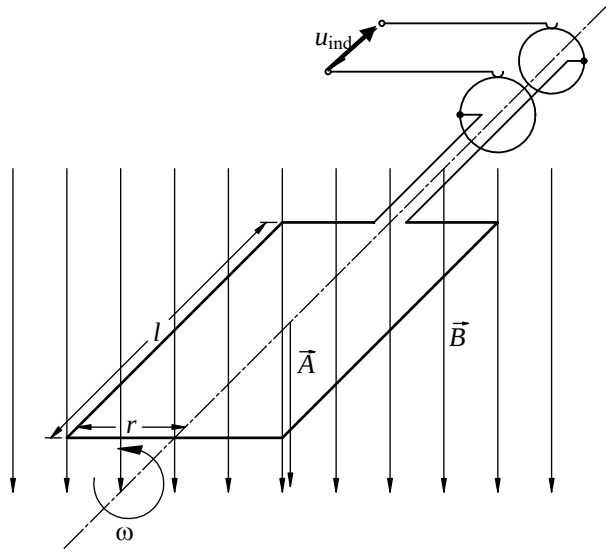
A transformer is not usually realized with only one conductor loop, but with a multitude of windings. In this case, the term flux of a conductor loop is replaced by the term flux linkage of a coil with N windings:

$$\Psi = N \cdot \Phi$$

In this case, the law of induction is deemed as follows:

$$u_{ind} = -N \cdot \frac{d\Phi}{dt} = -\frac{d\Psi}{dt}$$

10.1.4 Rotation of a conductor loop in the magnetic field



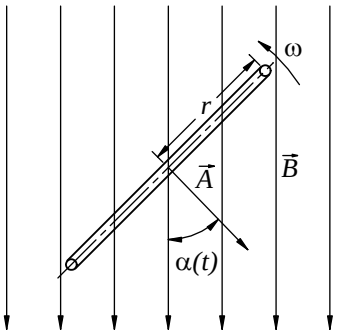
We assume a homogenous field with clear borders.

The magnetic flux which penetrates the conductor loop is:

$$\Phi = B \cdot l \cdot 2 \cdot r \cdot \cos(\alpha)$$

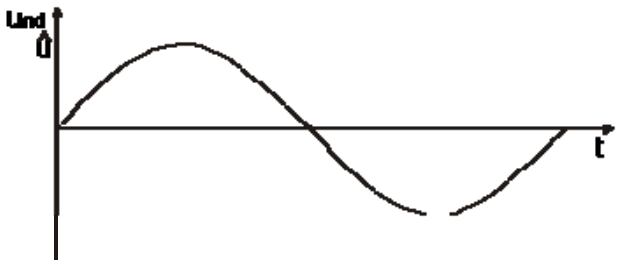
Uniform rotation: $\alpha = \omega \cdot t$

$$\Phi = B \cdot l \cdot 2 \cdot r \cdot \cos(\omega \cdot t)$$



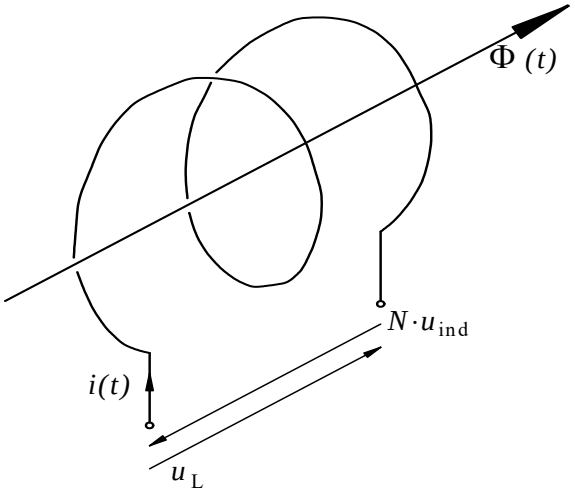
A voltage is induced into the conductor loop.

$$\begin{aligned} u_{ind} &= -\frac{d\Phi}{dt} \\ u_{ind} &= -B \cdot l \cdot 2 \cdot r \cdot (-\omega \cdot \sin(\omega \cdot t)) \\ u_{ind} &= \underbrace{2 \cdot r \cdot l \cdot B \cdot \omega}_{\hat{u}} \cdot \sin(\omega \cdot t) \\ \underline{u_{ind}} &= \underline{\hat{u} \cdot \sin(\omega \cdot t)} \end{aligned}$$



10.2 Inductance

10.2.1 Self-induction



When several series conductor loops (a coil) are penetrated by the same time-variable flux Φ(t), the voltage is induced in each winding:

$$u_{ind.} = -\frac{d\Phi}{dt}$$

altogether therefore:

$$u_{ind,ges} = -N \cdot \frac{d\Phi}{dt}$$

N = 2

Especially in the electrical power engineering this situation is often described as follows:

A Coil with N windings, cross section area A, spans N - times the area A, which is penetrated by the flux Φ each time. The overall active flux Ψ, the so-called linked magnetic flux amounts to:

$$\Psi = N \cdot \Phi \qquad \text{and} \qquad u_{ind,ges} = -\frac{d\Psi}{dt}$$

To generate a magnetic flux Φ(t), it is sufficient to send an alternating current (in the marked direction) through the coil. For a coil (passive component), the user related arrow system is more suitable $\rightarrow$ introduction of the voltage $u_L = -u_{ind,ges}$:

$$u_L = N \cdot \frac{d\Phi}{dt} = \frac{d}{dt}\left(N \cdot \Phi\right) \qquad \Phi = \frac{\Theta}{R_m} = \frac{N \cdot i}{R_m}$$

$$u_L = \frac{d}{dt}\left(\frac{N^2}{R_m} \cdot i\right)$$

$$\underline{\underline{u_L = \frac{d}{dt}\left(L \cdot i\right)}} \qquad \boxed{L = \frac{N^2}{R_m}} \text{ Self-inductance}$$

If $R_m = f(i)$, it is deemed due to $i = f(t)$ that also $L = f(t)$!

$$\frac{d}{dt}(N \cdot \Phi) = \frac{d}{dt}(L \cdot i)$$

$$\underline{\underline{N \cdot \Phi = L \cdot i}} \quad + k$$

(Integration constant $k = 0$, then $\Phi = 0$, if $i = 0$)

Otherwise, for example air coil (usual):

$$\underline{\underline{u_L = L \cdot \frac{di}{dt}}}$$

$\text{Unit}[L] = \frac{Vs}{A} = H$ (Henry) graphical symbol:



Let's recapitulate:

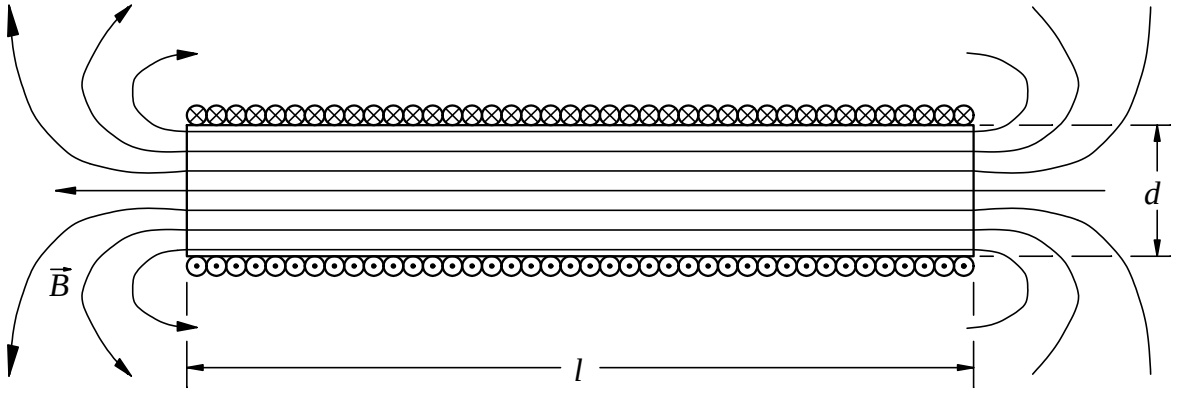
- The inductance is an integral parameter of the magnetic field, similar to the capacitance of the electrical field.
- The self-inductance indicates which magnetic flux a given coil current causes in the coil itself. This equation can also be used as a calculation rule for the calculation of an inductance. This calculation is useful especially when the flux of the coil can be measured almost exactly with the aid of a numerical program for the calculation of magnetic fields.

$$L = \frac{\Psi}{i} = \frac{N \cdot \Phi}{i}$$

- The self-inductance of a coil is proportional to the square of the number of turns per unit length of the coil. Furthermore the inductance is defined by the environment in which the magnetic field of the coil is located. This is expressed by the magnetic resistance R_m. An air gap in a magnetic circuit, for example, signifies a noticeable enlargement of the magnetic resistance and therefore a reduction of the self-inductance.

$$L = \frac{N \cdot \Phi}{i} = N^2 \cdot \Lambda = \frac{N^2}{R_m}$$

10.2.2 Self-inductance of the long cylinder coil



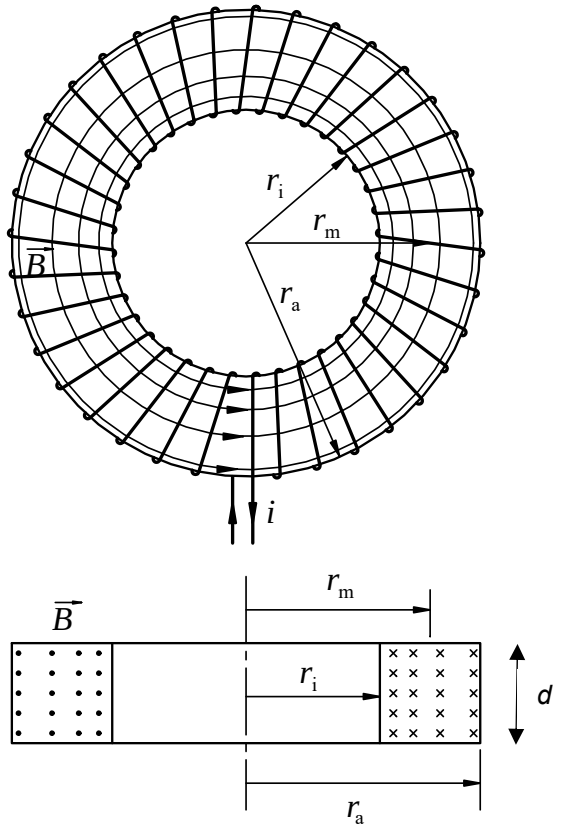
The magnetic resistance of the outer space is negligibly small compared to the resistance of the coil core (compare section 9.2.6)

"long" $\Rightarrow l >> d$

$$L = \frac{N^2}{R_m} \quad \text{with} \quad R_m \approx \frac{l}{\mu \cdot A}$$

$$\underline{\underline{L = N^2 \cdot \mu \cdot \frac{A}{l}}}$$

10.2.3 Self-inductance of the ring coil without air gap



a) approximate cálculation

$$r_i \approx r_a \approx r_m = \frac{r_a + r_i}{2}$$

The long ring coil can be seen as a long cylinder coil without outside space that is curved with its ends touching:

$$L \approx N^2 \cdot \mu \cdot \frac{A}{2\pi \cdot r_m} \text{ with } A = (r_a - r_i) \cdot d$$

$$\underline{\underline{L \approx N^2 \cdot \mu \cdot \frac{d}{\pi} \cdot \frac{r_a - r_i}{r_a + r_i}}}$$

b) Exact calculation

$$\Theta = \oint \vec{H} \cdot d\vec{s} = H \cdot 2\pi \cdot r \qquad L = \frac{N^2}{R_m} = \frac{N^2 \cdot \Phi}{\Theta}$$

$$\rightarrow H = \frac{\Theta}{2\pi \cdot r}$$

$$\Phi = \int_A B \cdot dA = \int_{r_i}^{r_a} \mu \cdot H \cdot d \cdot dr$$

$$\Phi = \mu \cdot d \cdot \int_{r_i}^{r_a} \frac{\Theta}{2\pi \cdot r} \cdot dr$$

$$\Phi = \mu \cdot d \cdot \frac{\Theta}{2\pi} \cdot \ln\left(\frac{r_a}{r_i}\right) \qquad L = N^2 \cdot \mu \cdot \frac{d}{2\pi} \cdot \ln\left(\frac{r_a}{r_i}\right)$$

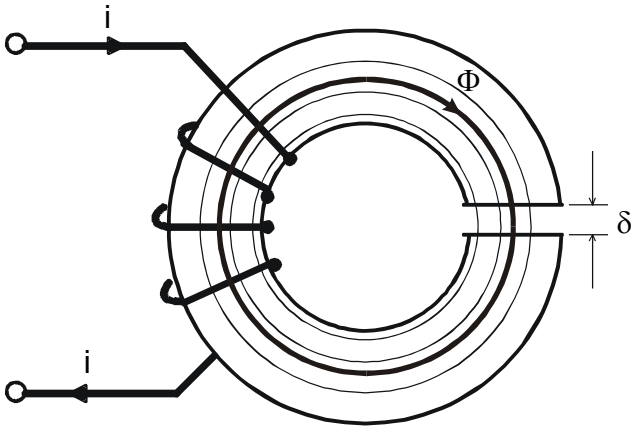
The solution of the approximation leads to slightly smaller results.
The difference is < 4%, if $r_a/r_i < 2$.

10.2.4 Self-inductance of the ring coil with air gap

An exact calculation is not possible anymore; but because of the good results of the approximation no longer necessary. The calculation is therefore only approximately correct because the magnetic resistance is calculated as if the magnetic field were homogenous. As a rule, the length of the centre line is usually substituted for the length l.

Calculation is done with:

- median iron path length
- $B_L = B_E$ (which means by neglecting the dispersion at the air gap)



Initial equation:

$$L = \frac{N^2}{R_m}$$

Analogue to the electrical resistance:

$$R = \frac{l}{\kappa \cdot A} \rightarrow R_m = \frac{l}{\mu \cdot A}$$

The total magnetic resistance of the magnetic circuit results in:

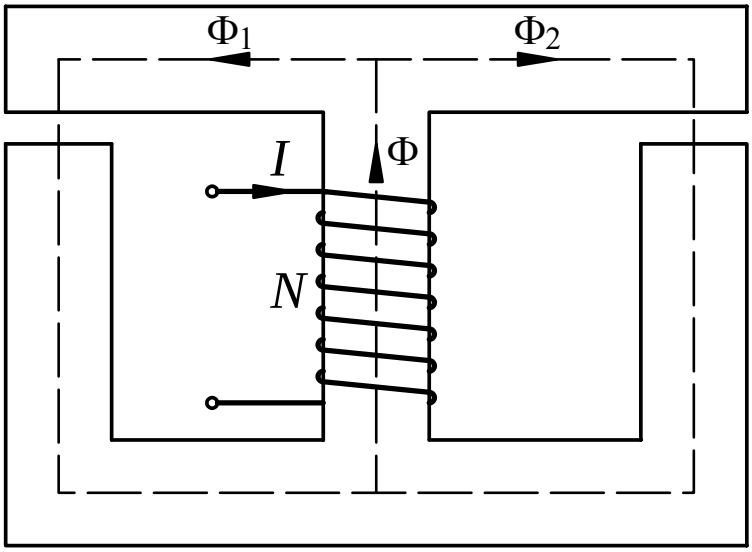
$$R_m = R_{mE} + R_{mL}$$

$$R_m = \frac{l}{\mu_r \cdot \mu_0 \cdot A} + \frac{\delta}{\mu_0 \cdot A} = \frac{1}{\mu_0 \cdot A} \cdot \left(\frac{l}{\mu_r} + \delta \right)$$

$$L = \frac{N^2 \cdot \mu_0 \cdot A}{\frac{l}{\mu_r} + \delta}$$

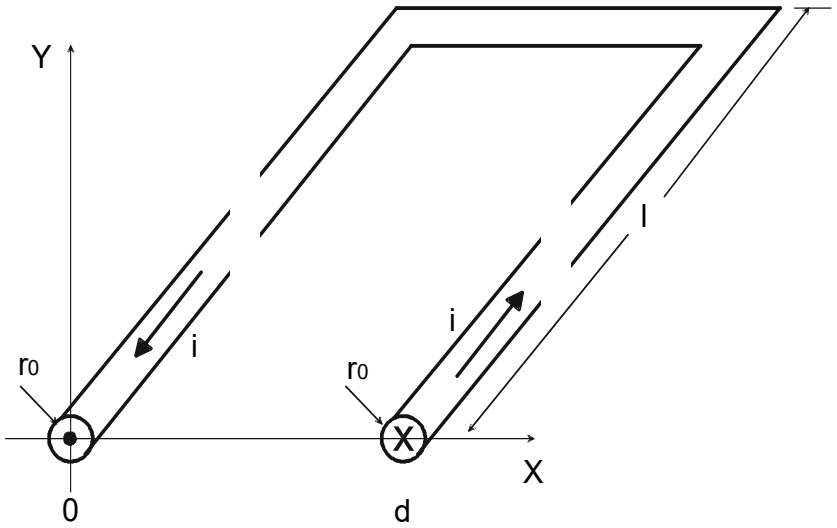
This method of determining the self-inductance with the magnetic resistance is also applicable to complicated iron core structures (see exercises).

Magnetic iron circuits and their self-inductance are actually computed this way, too.



10.2.5 Self-inductance of the long double-wire circuit

Initially we calculate only the magnetic field between the conductors of a double-wire circuit. Thereof the so-called outer inductance can be calculated.



Outer Inductance L_a

$L \approx L_a$, when $d >> r_0$

$L_a \cdot i = N \cdot \Phi_a \quad \text{here}: N = 1$

$$L_a = \frac{\Phi_a}{i}$$

The magnetic field has only a y - component in the plane y = 0.

$$H = H_y = \frac{i}{2\pi \cdot x} + \frac{i}{2\pi \cdot (d - x)}$$

$$\begin{aligned}
\Phi_a &= \int B \cdot dA = \int_{x=r_0}^{d-r_0} \mu_0 \cdot H \cdot l \cdot dx \\
\Phi_a &= \int_{x=r_0}^{d-r_0} \mu_0 \cdot l \cdot \frac{i}{2\pi} \cdot \left(\frac{1}{x} + \frac{1}{d-x} \right) \cdot dx \\
\Phi_a &= \frac{\mu_0 \cdot l \cdot i}{2\pi} \cdot \left[\ln(x) - \ln(d-x) \right]_{r_0}^{d-r_0} \\
\Phi_a &= \frac{\mu_0 \cdot l \cdot i}{2\pi} \cdot \left[\ln(d-r_0) - \ln(r_0) - \ln(r_0) + \ln(d-r_0) \right] \\
\Phi_a &= \frac{\mu_0 \cdot l \cdot i}{\pi} \cdot \ln\left(\frac{d-r_0}{r_0} \right)
\end{aligned}$$

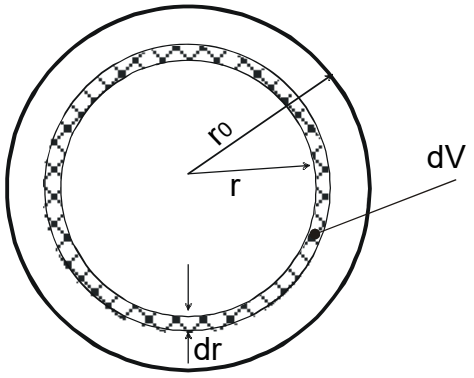
$$\underline{\underline{L_a = \frac{\mu_0 \cdot l}{\pi} \cdot \ln\left(\frac{d-r_0}{r_0} \right) \approx \frac{\mu_0 \cdot l}{\pi} \cdot \ln\left(\frac{d}{r_0} \right)}}$$

The outer inductance of a double-wire circuit can be reduced by decreasing the distance:

$$d \to 2r_0 \;\to\; L_a = \frac{\mu_0 \cdot l}{\pi} \cdot \ln\left(\frac{2 \cdot r_0 - r_0}{r_0} \right) = 0$$

Indeed, even with $d = 2r_0$ the double-wire circuit has an inner inductance L_i, because magnetic fields are also present within the conductor.

The inner inductance is calculated using the energy density (section 10.3.2)



$$\begin{aligned}
W_m &= \int w_m \cdot dV \\
\frac{1}{2} L \cdot i^2 &= \int_0^{r_0} \frac{1}{2} \cdot \mu \cdot H^2 \cdot l \cdot 2\pi \cdot r \cdot dr
\end{aligned}$$

$$H = \frac{i \cdot r}{2\pi \cdot r_0^{\,2}} \quad \text{(see 1.2.3)}$$

$$\begin{aligned}
L \cdot i^2 &= \int_0^{r_0} \mu \cdot l \cdot 2\pi \cdot r \cdot \frac{i^2 \cdot r^2}{(2\pi)^2 \cdot r_0^{\,4}} \cdot dr \\
L &= \frac{\mu \cdot l}{2\pi \cdot r_0^{\,4}} \cdot \int_0^{r_0} r^3 \cdot dr \\
L &= \frac{\mu \cdot l}{8\pi}
\end{aligned}$$

For a wire with the total length $2 \cdot l$ (forward and return conductor) the result of the inner inductance is:

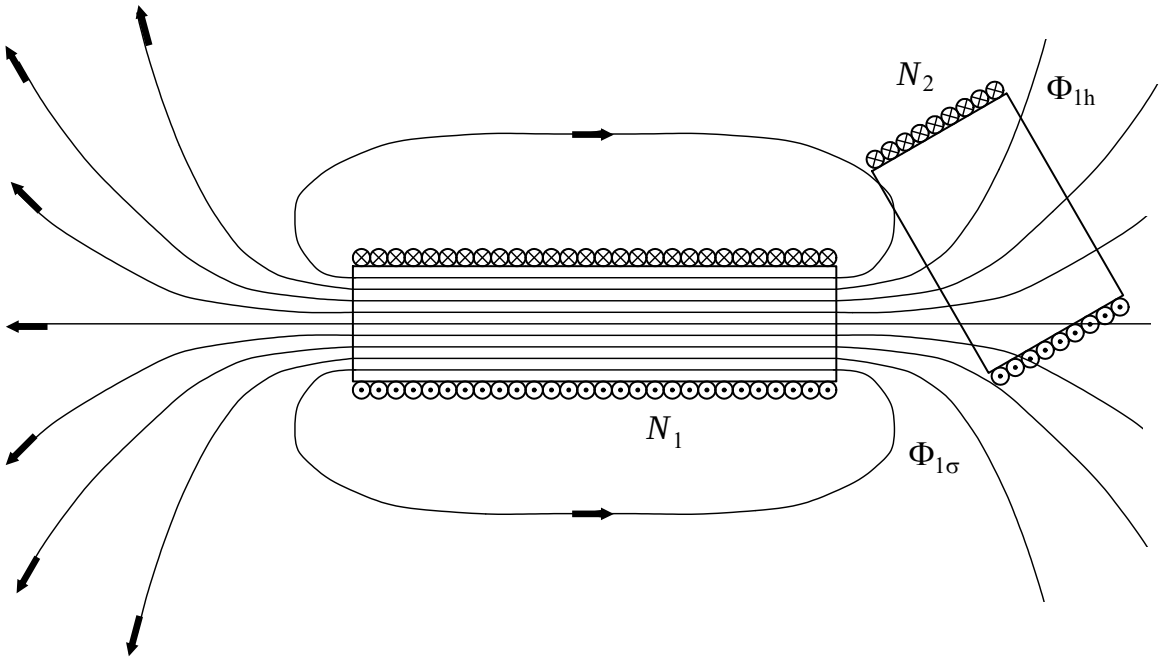
$$L_i = \frac{\mu \cdot l}{4\pi} \quad \text{with} \quad \mu = \mu_r \cdot \mu_0$$

(especially significant if the wire is made from iron.)

$$L = L_a + L_i = \frac{\mu_0 \cdot l}{\pi} \cdot \left[\ln\left(\frac{d - r_0}{r_0} \right) + \frac{\mu_r}{4} \right]$$

10.2.6 Mutual inductance

Pictured are two electric circuits that shall keep their mutual assemblies unaltered. The first circuit is electrified (i.e. a current flows); the second one isn't. A magnetic flux Φ_1 is generated whose size is proportional to the current i_1. A part Φ_{1h} of this flux penetrates the second circuit. This part of the flux is also proportional to the current i_1. Another part of the flux doesn't reach the second circuit and is therefore not linked with it. This part of the current created by the first circuit is identified as leakage flux $\Phi_{1\sigma}$. Depending on the arrangement of the two circuits and the medium in which the magnetic fields of these circuits are located only a portion of the flux will ever contribute to the coupling of the two circuits. This coupling can now be described with another integral parameter of the magnetic field. We identify this parameter as the mutual inductance M. It indicates which flux linkage is caused by current i_1 in the second coil.



Two neighbouring coils are electrified and (partially) penetrate each other with their magnetic fluxes.

Application of the superposition principle:

$i_2 = 0$ (see picture)			$i_1 = 0$ (w/o picture)		
$\Phi_1 = \frac{N_1 \cdot i_1}{R_{m1}}$			$\Phi_2 = \frac{N_2 \cdot i_2}{R_{m2}}$		
$\Phi_{1h} = k_1 \cdot \frac{N_1 \cdot i_1}{R_{m1}}$		main flux	$\Phi_{2h} = k_2 \cdot \frac{N_2 \cdot i_2}{R_{m2}}$		
$\Phi_{1\sigma} = (1-k_1) \cdot \frac{N_1 \cdot i_1}{R_{m1}}$		leakage flux	$\Phi_{2\sigma} = (1-k_2) \cdot \frac{N_2 \cdot i_2}{R_{m2}}$		
$u_2' = -N_2 \cdot u'_{ind2}$			$u_1'' = -N_1 \cdot u''_{ind1}$		Allocation of the current arrow to the flux arrow as for the self-inductance. Hence the negative sign.
$u_2' = N_2 \cdot \frac{d\Phi_{1h}}{dt}$			$u_1'' = N_1 \cdot \frac{d\Phi_{2h}}{dt}$		
$u_2' = N_2 \cdot k_1 \cdot \frac{N_1}{R_{m1}} \cdot \frac{di_1}{dt}$			$u_1'' = N_1 \cdot k_2 \cdot \frac{N_2}{R_{m2}} \cdot \frac{di_2}{dt}$		
$u_2' = M_{12} \cdot \frac{di_1}{dt}$			$u_1'' = M_{21} \cdot \frac{di_2}{dt}$		

$$M = M_{12} = M_{21} = k_1 \cdot \frac{N_1 \cdot N_2}{R_{m1}} = k_2 \cdot \frac{N_1 \cdot N_2}{R_{m2}}$$ Mutual inductance

$$(\text{without evidence}: \frac{k_1}{R_{m1}} = \frac{k_2}{R_{m2}} \text{ , results from the energy balance})$$

It can be shown that the value of the mutual inductance M is not dependent on the fact, which one of the two circuits is electrified (i.e carries a current). The only basic requirement is that it is a linear medium.

For the coil's self-induced voltage, it is still deemed:

$$u_1' = L_1 \cdot \frac{di_1}{dt} \qquad u_2'' = L_2 \cdot \frac{di_2}{dt}$$

Superposition:

$$u_1 = u_1' + u_1''$$
$$u_2 = u_2' + u_2''$$

$$\boxed{\begin{aligned} u_1 &= L_1 \cdot \frac{di_1}{dt} + M \cdot \frac{di_2}{dt} \\ u_2 &= M \cdot \frac{di_1}{dt} + L_2 \cdot \frac{di_2}{dt} \end{aligned}}$$

Coupling factors: $k_1 = \frac{\Phi_{1h}}{\Phi_1}$ $k_2 = \frac{\Phi_{2h}}{\Phi_2}$

(Total) coupling factor $k = \sqrt{k_1 \cdot k_2}$

Technical transmitters and transformers are (generally) assembled in such a way that the coupling factor and the magnetic resistance for both windings are the same.

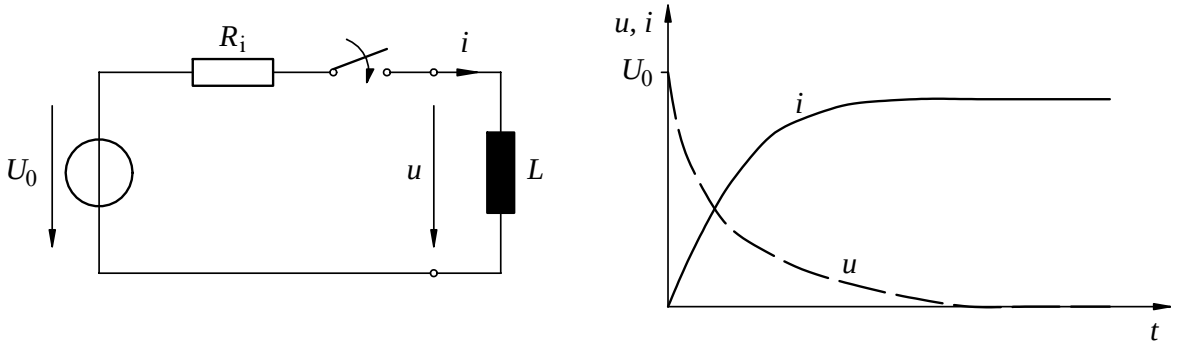
$k_1 \approx k_2 \approx k$ $R_{m1} \approx R_{m2} \approx R_m$

Correlation between self-inductance and mutual inductance:

$$\left.\begin{aligned} L_1 &= \frac{N_1^2}{R_m} \\ L_2 &= \frac{N_2^2}{R_m} \end{aligned}\right\} \quad \left.\begin{aligned} L_1 \cdot L_2 &= \frac{N_1^2 \cdot N_2^2}{R_m^2} \\ M &= k \cdot \frac{N_1 \cdot N_2}{R_m} \Rightarrow \\ M^2 &= k^2 \cdot \frac{N_1^2 \cdot N_2^2}{R_m^2} \end{aligned}\right\} \quad \left.\begin{aligned} M^2 &= k^2 \cdot L_1 \cdot L_2 \\ \boxed{M = k \cdot \sqrt{L_1 \cdot L_2}} \end{aligned}\right\}$$

10.3 Energy of the magnetic field

10.3.1 Energy of the current-carrying coil



The energy which is stored in a coil can be calculated with the aid of instantaneous power. After closing the switch, energy is transported to the coil:

- At the time t = 0 the switch is closed.
- At the time t = t_1 the stationary condition is reached.

$0 < t < t_1$ $t > t_1$

$i = i(t)$
$u = u(t)$
$\Phi = \Phi(t)$

$$u = L \cdot \frac{di}{dt} = N \cdot \frac{d\Phi}{dt}$$

$i = I$
$u = 0$
$\Phi = const$

$$W_m = \int_0^{t_1} i \cdot u \cdot dt$$

L = konst.

$$W_m = L \cdot \int_0^{t_1} i \cdot \frac{di}{dt} \cdot dt$$

$$W_m = L \cdot \int_0^{I} i \cdot di$$

$$\boxed{W_m = \frac{1}{2} \cdot L \cdot I^2}$$

L ≠ konst.

$$W_m = \int_0^{t_1} i \cdot N \cdot \frac{d\Phi}{dt} \cdot dt$$

$$\boxed{W_m = \int_0^{\Phi} \Theta \cdot d\Phi}$$

First, we shall further examine the left side. The energy stored in the coil does not depend on the time slope of the current, but only on the value of the inductance and the square of the current which is actually flowing in the coil.

We already know a similar formula from the electrical field:

$$W_e = \frac{1}{2} \cdot C \cdot U^2$$

10.3.2 Energy density

The energy which is stored in a coil can also be expressed with the help of the field quantities B and H. Hereby we encounter the quantity energy density.

For that we now examine the right side of the equation shown in 10.3.1. In the homogenous field of a long cylindrical coil, it is deemed:

$$\left.\begin{aligned} \Theta &= H \cdot l \\ \Phi &= B \cdot A \end{aligned}\right\} \quad W_m = \int_0^{B \cdot A} H \cdot l \cdot d(B \cdot A) = A \cdot l \cdot \int_0^{B} H \cdot dB$$

$$\frac{W_m}{V} = w_m = \int_0^{B} H \cdot dB$$

Infinitesimally small portions of inhomogeneous fields are quasi homogenous

$$\boxed{\frac{dW_m}{dV} = w_m = \int_0^{B} H \cdot dB}$$

If $\mu = \text{constant}$

it is deemed $\int_0^{B} H \cdot dB = \int_0^{B} \frac{B}{\mu} \cdot dB = \frac{B^2}{2 \cdot \mu}$

$$w_m = \frac{1}{2} \cdot B \cdot H = \frac{1}{2} \cdot \mu \cdot H^2 = \frac{B^2}{2 \cdot \mu}$$

Here, a similar relationship is also known from the electrical field:

$$\frac{dW_e}{dV} = w_e = \frac{1}{2} \cdot D \cdot E = \frac{1}{2} \cdot \varepsilon \cdot E^2 = \frac{D^2}{2 \cdot \varepsilon}$$

The energy density is often used to describe the energy in a definite volume element of the magnetic field. The correlation between the two quantities is generally given by a volume integral:

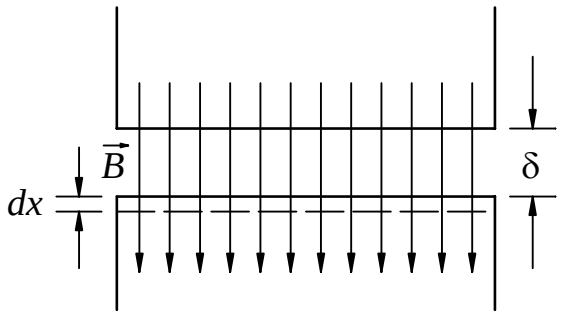
$$W_m = \int_V w_m \cdot dV = \int_V \frac{\vec{B} \cdot \vec{H}}{2} \cdot dV$$

For a system with two electrical circuits (compare to section 10.2.6) the following formula to calculate the magnetic energy is deemed:

$$W_m = \frac{1}{2} \cdot L_1 \cdot I_1^2 + \frac{1}{2} \cdot L_2 \cdot I_2^2 + M \cdot I_1 \cdot I_2$$

10.3.3 Forces on magnetic pole areas

As in the electrical field, the forces in a magnetic field can be calculated using the energy change. Here a virtual shift of a portion of the system is carried out and the change of the magnetic field energy is calculated. This energy change directly results in the force effect on the body which was virtually shifted.



The dispersion at the air gap is neglected:

$$B = B_E = B_L \quad \text{(homogenous field)}$$

The energy stored in the air gap is:

$$W_m = \int_V w_m \cdot dV = \frac{B^2}{2 \cdot \mu_0} \cdot A \cdot \delta$$

An increase of the air gap to $\delta + dx$ causes an increase of the stored energy to $W_m + dW_m$.

A change of the energy stored in the iron does not happen as long as B doesn't change (because of the increase of R_m).

$$W_m + dW_m = \frac{B^2}{2 \cdot \mu_0} \cdot A \cdot (\delta + dx)$$

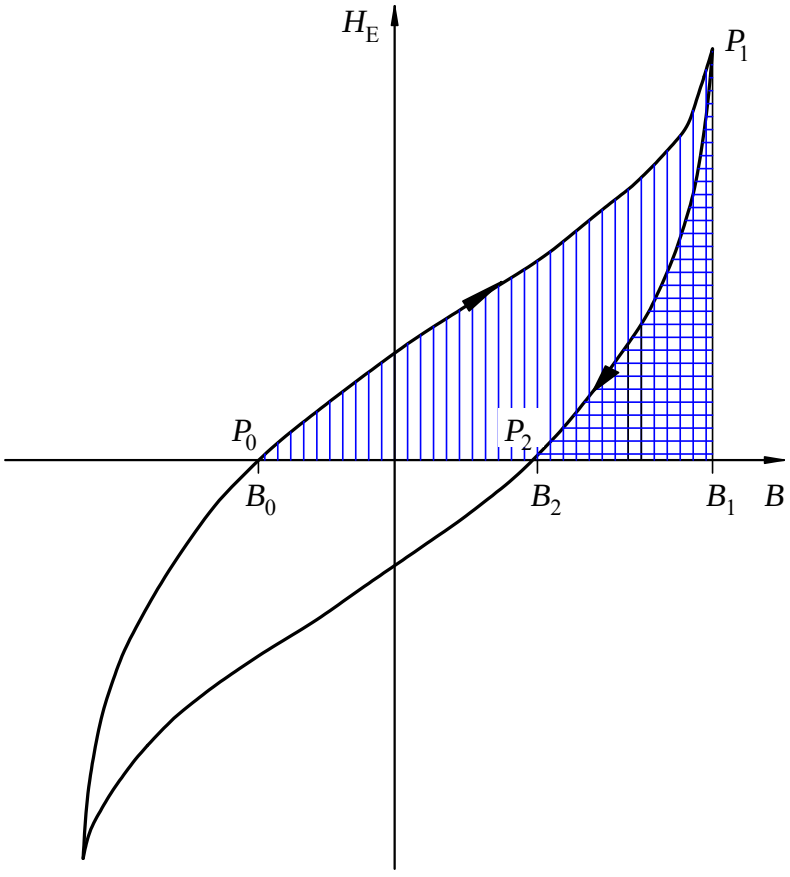
For the mechanically necessary work, it is deemed:

$$dW = F \cdot dx \qquad dW_m = \frac{B^2}{2 \cdot \mu_0} \cdot A \cdot dx$$

$$\boxed{F = \frac{B^2}{2 \cdot \mu_0} \cdot A} \qquad F \sim B^2$$

as opposed to $\vec{F} = I \cdot \vec{l} \times \vec{B}$ $(F \sim B)$

10.3.4 Hysteresis losses



Section 10.3.2:

$$w_m = \int_0^B H \cdot dB$$

$$w_m\big|_{Eisen} = \int H_E \cdot dB$$

$$w_m\big|_{Luft} = \frac{B^2}{2 \cdot \mu_0}$$

$$W_m = \int_V w_m \cdot dV$$

W_{01} is the energy which has to be expended in order to change the magnetic state of iron from point P_0 to point P_1:
The vertical-shaded area under the curve is a measure of the energy which is added to the iron.

$$W_{01} = A \cdot l \cdot \int_{B_0}^{B_1} H_E \cdot dB + A \cdot \delta \cdot \frac{B_1^{\,2} - B_0^{\,2}}{2 \cdot \mu_0}$$

Shifting the bias point from P_1 to P_2 results accordingly in:
The horizontal-shaded area under the curve is a measure for the energy which is regained from the iron.

$$W_{12} = A \cdot l \cdot \int_{B_1}^{B_2} H_E \cdot dB + A \cdot \delta \cdot \frac{B_2^{\,2} - B_1^{\,2}}{2 \cdot \mu_0}$$

The energy which is added to the air gap is regained completely; because of
$B_0^{\,2} = B_2^{\,2}$

The energy $W_{01} - W_{12}$ is lost
$\rightarrow$ rise in the temperature of the iron

The hysteresis losses are therefore proportional to the area of the hysteresis loop and reoccur each cycle, thus it is deemed:

Hysteresis power dissipation is proportional to the frequency

(The eddy current power dissipation, however, is ~ f^2, because $P = \frac{u^2}{R}$ and $u \sim \frac{d\Phi}{dt} \sim f$)

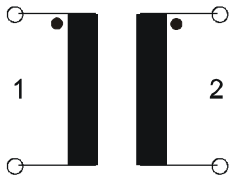
10.4 Transformers and transmitters

Transformer: electrical power technology
Transmitter: information technology

Goal:

- serves for stepping (transforming) voltages and currents up and down
- galvanic isolation (isolating transformer)

graphical symbol:

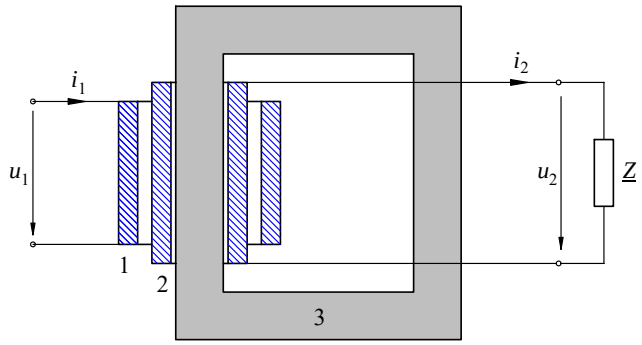


1: Primary winding = input side, this means the energy feed is here, where the voltage source is connected
2: Secondary winding = output side, this means the energy is extracted here

- Winding direction (here, the same)

There are many types of transformers. One example is a so-called two-winding transformer.

10.4.1 Ideal transformer



1: Primary winding - winding number N_1

2: Secondary winding - winding number N_2

3: Iron core

Requirements for the ideal transformer:

- Both coil windings are wound in the same direction and without losses: $R_1 = 0$, $R_2 = 0$.
- The iron core is "ideal", this means the permeability $\mu_r \to \infty$ and no eddy current or magnetic reversal losses should occur. Thereby, it presents an infinite magnetic conductivity $\Lambda \to \infty$, respectively $R_m = 0$.
- Both windings are penetrated by the same magnetic flux Φ, thus no dispersion occurs.

Law of induction

Since the primary and secondary coils are wound in the same direction and penetrated by the same magnetic flux, the two following voltages appear on the clamps of the two windings according to the law of induction:

$$u_1 = -N_1 \cdot \frac{d\Phi}{dt} \qquad\qquad u_2 = -N_2 \cdot \frac{d\Phi}{dt}$$

Φ respectively $\frac{d\Phi}{dt}$ are the same in both windings

$$\Rightarrow \quad \boxed{\frac{u_1}{u_2} = \frac{U_1}{U_2} = \frac{N_1}{N_2} = \ddot{u}} \qquad \ddot{u} = \text{transformation ratio}$$

The transformation ratio applies to the time values u_1 and u_2 of the voltages as well as to the root mean squares U_1 and U_2.

Ampere's law

Since the magnetic resistance of the magnetic circuit is $R_m = 0$, the current linkage must be 0, according to Ohm's law of the magnetic circuit (section 9.5.5):

$$\Theta = R_m \cdot \Phi = 0$$

As per the count arrow system used in the picture, the currents i_1 and i_2, which appear in coils with the same winding direction, flow in opposite directions. For the current linkage it must be deemed:

$$\Theta = N_1 \cdot i_1 - N_2 \cdot i_2 = 0$$

$$\Rightarrow \quad \boxed{\frac{i_1}{i_2} = \frac{I_1}{I_2} = \frac{N_2}{N_1} = \frac{1}{\ddot{u}}}$$

The currents are transformed in an inverse proportion to the voltages: the voltage is transformed up to the higher winding number, the current is transformed down at the same ratio.

therefore: $\boxed{\ddot{u} = \frac{N_1}{N_2} = \frac{U_1}{U_2} = \frac{I_2}{I_1}}$

Power view

According to the transformation ratio, the two voltages u_1 and u_2 as well as the two currents i_1 and i_2 are in phase with each other. On both the primary and secondary sides, the same phase angle φ exists between current and voltage, which is determined on the secondary side, for example by the impedance $\underline{Z}_a$. For the apparent power it can be shown:

$$S_1 = U_1 \cdot I_1 = \frac{N_1}{N_2} \cdot U_2 \cdot \frac{N_2}{N_1} \cdot I_2 = S_2$$

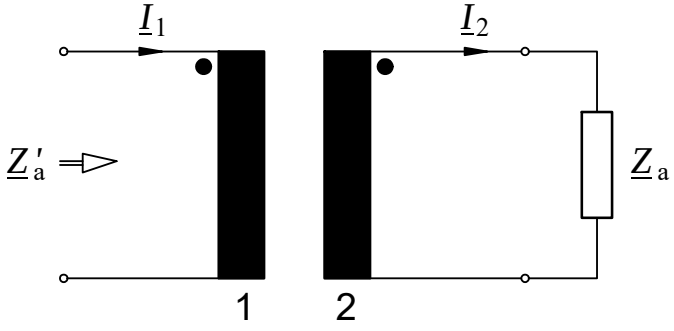
Since, per definition, no losses can occur in the ideal transformer, the primarily-absorbed and secondarily-delivered effective power must also be the same: $P_1 = P_2$.

A corresponding result must then ultimately apply to the reactive power: $Q_1 = Q_2$

These power transmission characteristics also apply to technical transformers in the first approximation. In electrical power engineering for a long-distance power transmission, for example, voltage is stepped up to high values (e.g. 400 kV) so that the currents can be simultaneously stepped down to small values, thereby reducing the power losses. Before reaching the consumer, the voltage must then be stepped down to the nominal low voltage of the consumer power grid (e.g. 230 V).

Transformation of impedances

Ideal transformer with load



$$\underline{S}_2 = \underline{Z}_a \cdot I_2^{\,2}$$

Question: Which load, which impedance results on the primary side?

$$\underline{S}_1 = \underline{Z}_a' \cdot I_1^{\,2}$$

$$\underline{S}_1 = \underline{S}_2 \;\Rightarrow\; \underline{Z}_a' = \frac{I_2^{\,2}}{I_1^{\,2}} \cdot \underline{Z}_a = \ddot{u}^2 \cdot \underline{Z}_a$$

Impedances are therefore squarely stepped up to the higher winding number using the transformation ratio. This possibility often is used in information technology, to transmit the values of impedances of a receiver (almost loss free, which means without pre-resistor and voltage divider), e.g. to reach power matching (hence the name transmitter).

The secondary electrical quantities of a transformer which is built into a network can be converted to the primary side (indicator: ′)

$$\underline{U}_2 \;\Rightarrow\; \underline{U}_2' = \ddot{u} \cdot \underline{U}_2$$
$$\underline{I}_2 \;\Rightarrow\; \underline{I}_2' = \frac{1}{\ddot{u}} \cdot \underline{I}_2$$
$$\underline{Z}_a \;\Rightarrow\; \underline{Z}_a' = \ddot{u}^2 \cdot \underline{Z}_a$$

For the ideal transformer, this means:

$$I_1 = I_2'$$
$$U_1 = U_2'$$

Then, for the description of the behaviour of alternating current, the ideal transformer can be "removed" from the circuit. The circuit behaviours are, observed from the primary side, reproduced correctly (exception: the galvanic isolation!).

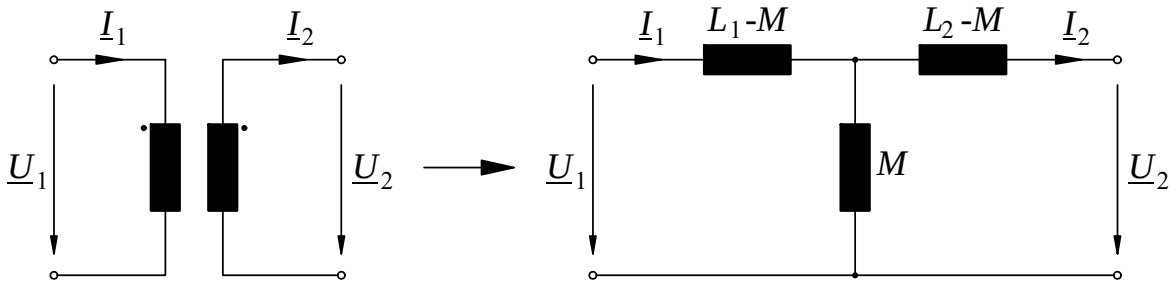
In an appropriate manner, a conversion of the primary quantities $\underline{U}_1, \underline{I}_1, \underline{Z}_1$ to the secondary side is possible.

10.4.2 Lossless transformer

The transformer is now only at zero loss but:

- finite magnetical conductvity of the iron core, which means $R_m > 0$
- dispersion present, which means $k < 1$

The mutual linkage of primary and secondary coils is described in the equivalent circuit by the size of the mutual inductance *M* (compare to 10.2.6). L_1 and L_2 are the self-inductances of the two coils.



System of equations:
(KVL equations)

$$u_1 = \left(L_1 - M\right) \cdot \frac{di_1}{dt} + M \cdot \frac{di_1}{dt} - M \cdot \frac{di_2}{dt}$$

$$u_2 = M \cdot \frac{di_1}{dt} - M \cdot \frac{di_2}{dt} - \left(L_2 - M\right) \cdot \frac{di_2}{dt}$$

$$\boxed{\begin{aligned} u_1 &= L_1 \cdot \frac{di_1}{dt} - M \cdot \frac{di_2}{dt} \\ u_2 &= M \cdot \frac{di_1}{dt} - L_2 \cdot \frac{di_2}{dt} \end{aligned}}$$

This result conforms to the voltage equations from section 10.2.6 when the opposing directions of the current i_2 is considered. For sinusoidal quantities and after transfer to complex notations, it can then be written:

$$\boxed{\begin{aligned} \underline{U}_1 &= j\omega L_1 \cdot \underline{I}_1 - j\omega M \cdot \underline{I}_2 \\ \underline{U}_2 &= j\omega M \cdot \underline{I}_1 - j\omega L_2 \cdot \underline{I}_2 \end{aligned}}$$

Here it is also deemed: The equivalent circuit displays the properties of the zero loss transformer correctly (exception: galvanic isolation)

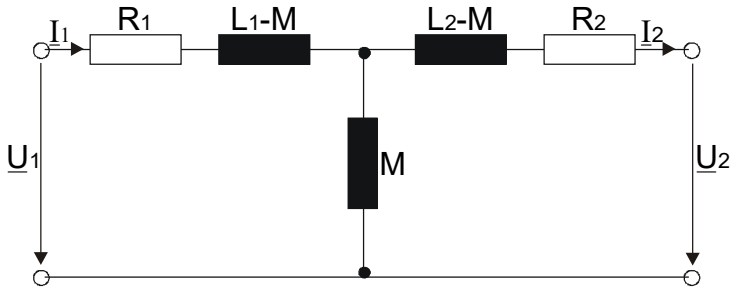
Attention: with $N_1 \neq N_2$ it can happen that either $\left(L_1 - M\right) < 0$ or $\left(L_2 - M\right) < 0$! Physically, this is not directly interpretable, the equivalent circuit cannot really be constructed this way, but it is correct for the calculation.

10.4.3 Transformer with copper losses

Generally, the power losses which occur in technical transformers can be divided into two categories:

- copper losses due to ohmic resistances of the (copper) windings
- core losses due to magnetization reversal and due to eddy currents in the (iron) core

Only copper losses are considered here, in the equivalent circuit simply by preconnecting the ohmic resistors R_1 and R_2 before the primary and secondary windings.



The arrangement of the KVL (Kirchhoff's voltage law) equation (analogue to 10.4.2) then leads to:

$$\begin{aligned}\underline{U}_1 &= R_1 \cdot \underline{I}_1 + j\omega\left(L_1 - M\right)\cdot \underline{I}_1 + j\omega M\left(\underline{I}_1 - \underline{I}_2\right)\\ \underline{U}_2 &= -R_2 \cdot \underline{I}_2 - j\omega\left(L_2 - M\right)\cdot \underline{I}_2 + j\omega M\left(\underline{I}_1 - \underline{I}_2\right)\end{aligned}$$

$$\boxed{\begin{aligned}\underline{U}_1 &= \left(R_1 + j\omega L_1\right)\cdot \underline{I}_1 - j\omega M \cdot \underline{I}_2\\ \underline{U}_2 &= j\omega M \cdot \underline{I}_1 - \left(R_2 + j\omega L_2\right)\cdot \underline{I}_2\end{aligned}}$$

If the parameters of the transformer are known, the system of equations still contains four unknown variables $\underline{U}_1$, $\underline{U}_2$, $\underline{I}_1$, $\underline{I}_2$. The load on the secondary side of the transformer can be described by:

$$\underline{U}_2 = \underline{Z}_2 \cdot \underline{I}_2$$

and if the input voltage $\underline{U}_1$ on the primary side is known; the system of equations can then be completely solved.

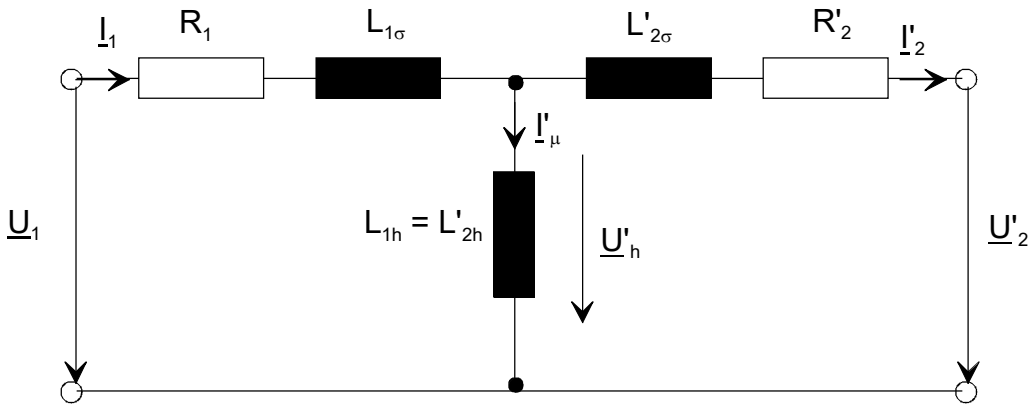
The secondary quantities can now be converted to the primary side again (compare to 10.4.1). Then the equivalent circuit can be better interpreted. For this purpose, the relationships $\underline{U}'_2 = \ddot{u}\cdot\underline{U}_2$ and $\underline{I}'_2 = \underline{I}_2/\ddot{u}$ are used. First, the two above-mentioned KVL equations are re-arranged:

$$\underline{U}_1 = R_1 \cdot \underline{I}_1 + j\omega\left(L_1 - \ddot{u}\cdot M\right)\cdot \underline{I}_1 + j\omega\cdot\ddot{u}\cdot M\left(\underline{I}_1 - \frac{1}{\ddot{u}}\cdot\underline{I}_2\right)$$

$$\ddot{u}\cdot\underline{U}_2 = -\ddot{u}^2\cdot R_2\cdot\frac{1}{\ddot{u}}\cdot\underline{I}_2 - j\omega\left(\ddot{u}^2\cdot L_2 - \ddot{u}\cdot M\right)\cdot\frac{1}{\ddot{u}}\cdot\underline{I}_2 + j\omega\cdot\ddot{u}\cdot M\left(\underline{I}_1 - \frac{1}{\ddot{u}}\cdot\underline{I}_2\right)$$

This results in the system of equations for the following equivalent circuit:

$$\begin{aligned}\underline{U}_1 &= R_1\cdot\underline{I}_1 + j\omega L_{1\sigma}\cdot\underline{I}_1 + j\omega L_{1h}\cdot\left(\underline{I}_1 - \underline{I}'_2\right)\\ \underline{U}'_2 &= -R'_2\cdot\underline{I}'_2 - j\omega L'_{2\sigma}\cdot\underline{I}'_2 + j\omega L'_{2h}\cdot\left(\underline{I}_1 - \underline{I}'_2\right)\end{aligned}$$



In the equivalent circuit there are:

- $L_{1\sigma} = L_1 - \ddot{u} \cdot M$ the primary leakage inductance;
- $L_{2\sigma} = L_2 - \frac{1}{\ddot{u}} \cdot M$ the secondary leakage inductance, which is converted to the primary side, of course: $L'_{2\sigma} = \ddot{u}^2 \cdot L_{2\sigma} = \ddot{u}^2 \cdot L_2 - \ddot{u} \cdot M$;
- $L_{1h} = \ddot{u} \cdot M$ the primary main inductance;
- $L_{2h} = \frac{1}{\ddot{u}} \cdot M$ the secondary main inductance, which is converted to the primary side again: $L'_{2h} = \ddot{u}^2 \cdot L_{2h} = \ddot{u} \cdot M$, and it naturally follows: $L_{1h} = L'_{2h}$.

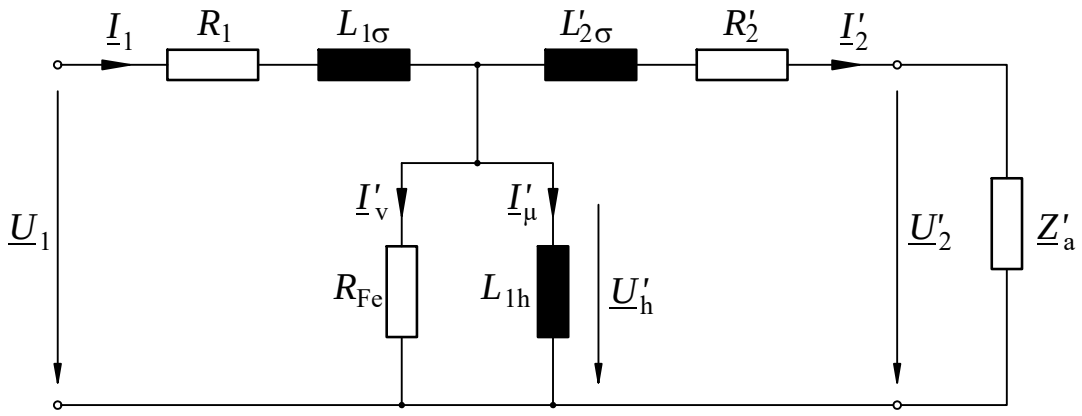
The magnetizing current which is converted to the primary side $\underline{I}'_{\mu} = \underline{I}_1 - \underline{I}'_2$ is the current necessary to generate the main flux (or utility flux) Φ_{1h}. The voltage $\underline{U}'_h$ is identified as main field voltage.

10.4.4 Transformers with copper and iron (core) losses

With technically-realized transformers, magnetization reversal and eddy current losses in the (iron) core as well as copper losses occur. These so-called core losses are approximately proportional to the square of the voltage and are considered in the equivalent circuit by an effective resistance R_{Fe} which lies parallel to the main inductance.

Besides the magnetizing current $\underline{I}'_{\mu}$ the effective portion $\underline{I}'_{v}$ is also obtained so that core losses can be described by the product $U'_h I'_v$.

With the help of the equivalent circuit which is presented here, it is now possible to describe the behaviour of a transformer with load.


$\underline{I}_1$
R_1
$L_{1\sigma}$
$L'_{2\sigma}$
R'_2
$\underline{I}'_2$
$\underline{U}_1$
$\underline{I}'_\mathrm{v}$
$\underline{I}'_\mu$
R_Fe
L_1h
$\underline{U}'_\mathrm{h}$
$\underline{U}'_2$
$\underline{Z}'_\mathrm{a}$

11. Maxwell's equations

This section serves as a summary of the fundamental laws of the electric and magnetic fields which have been derived so far (electrostatic field, current field, magnetic field). These correlations are identified as Maxwell's equations.

11.1 Ampere's law

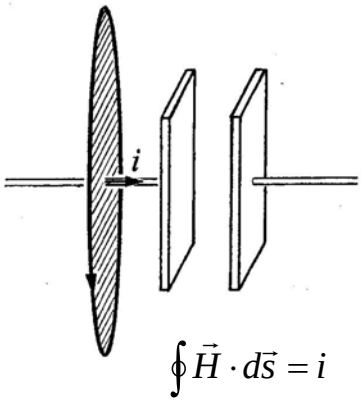
Ampere's law describes the correlation between the magnetic field strength and the current as the cause of this magnetic field. The application of this equation requires integration over a closed loop. The thereby enclosed current is identified as the current linkage. This law can be applied to the calculation of a magnetic field within a conductor. For stationary conditions the result therefore is:

$$\oint \vec{H} \cdot d\vec{s} = \Theta = \int \vec{J} \cdot d\vec{A}$$

With alternating fields, especially with high frequencies, the electric displacement current has to be taken into account. This is illustrated by the following example.

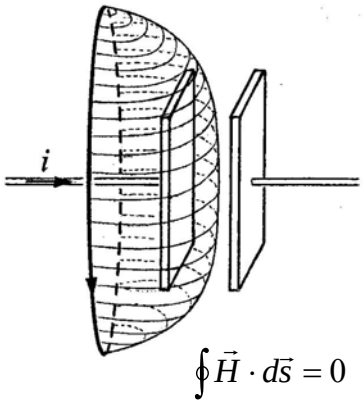
When the above-mentioned equation is applied to the transient case of a charged capacitor, two possible cases ensue:

area tightly stretched



$$\oint \vec{H} \cdot d\vec{s} = i$$

arched area guided through the capacitor plates



$$\oint \vec{H} \cdot d\vec{s} = 0$$

The integration path $\oint \vec{H} \cdot d\vec{s}$ is identical in both cases, however, the surface in the left case is electrified while no electric current *i* occurs in the surface on the right.

The second result is obviously wrong. Therefore a displacement current must be introduced into the area between the two capacitor plates, which results from the electric field:

$$i = \frac{dQ}{dt} \qquad Q = \oint \vec{D} \cdot d\vec{A} = \int \vec{D} \cdot d\vec{A}$$

$$i = \frac{dQ}{dt} = \frac{d}{dt}\int \vec{D} \cdot d\vec{A} = \int \frac{d\vec{D}}{dt} \cdot d\vec{A}$$ displacement current

with $\frac{d\vec{D}}{dt}$ = displacement current density

The general equation for Ampere's law (valid also for transient fields) reads as follows:

$$\oint \vec{H} \cdot d\vec{s} = \int \left(\vec{J} + \frac{d\vec{D}}{dt} \right) \cdot d\vec{A}$$

11.2 Faraday's law of induction

It is known from network analysis that in a single rotation on a closed mesh circuit the sum of all voltages is zero (Kirchhoff's mesh rule/KVL). Formulated for the electric field strength, this means:

$$\oint \vec{E} \cdot d\vec{s} = 0$$

This is true only when no time-variable magnetic fields occur. As we know, the voltage induced in a loop (mesh) results from the change of the magnetic flux which penetrates the loop:

$$u_{ind} = -\frac{d\Phi}{dt}$$

Therefore the general equation for the law of induction (valid also for transient fields) reads as follows:

$$\oint \vec{E} \cdot d\vec{s} = -\frac{d}{dt}\int \vec{B} \cdot d\vec{A}$$

11.3 Gauss's law

The integration of the electric flux density over a closed surface area results in the charge which is located within this surface area:

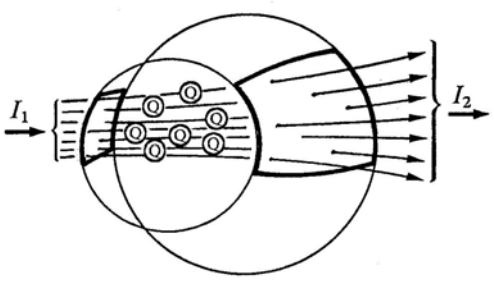
$$\oint \vec{D} \cdot d\vec{A} = Q$$

This relationship serves as the basis for the calculation of electric fields of various arrangements. If the charge *Q* is replaced by the volume integral over the space charge density ρ, then the general description ensues:

$$\oint \vec{D} \cdot d\vec{A} = \int \rho \cdot dV$$

11.4 Continuity equation

The first Kirchhoff law (KCL) states that the sum of all currents at a node results in zero. Formulated a bit differently, the stationary current field is source-free:



$$\oint \vec{J} \cdot d\vec{A} = 0$$

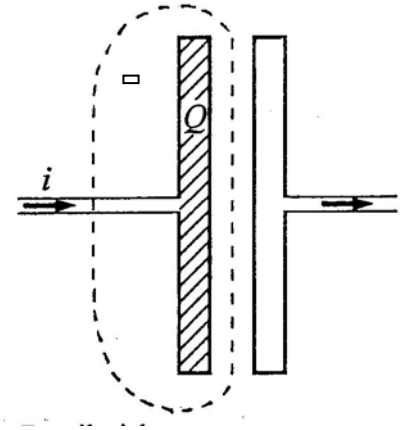
therefore: $I_2 - I_1 = 0$

$$I_2 = I_1$$

In contrast, it is possible that in a transient current field more (or fewer) charges temporarily flow into the volume than flow out.

Example: charging of a capacitor

Since no charge can be lost, this leads to an increase of charge (respectively a decrease) in the relevant volume:



$$i = \frac{dQ}{dt}$$

or more generally:

$$i_1 - i_2 = \frac{dQ}{dt} \text{ and } i_2 - i_1 = -\frac{dQ}{dt}$$

If the current difference is replaced by the integral of the current density over a closed surface area and the charge is replaced by the volume integral over the space charge density which is enclosed by the surface area, then the continuity equation follows:

$$\oint \vec{J} \cdot d\vec{A} = -\frac{d}{dt} \int \rho \cdot dV$$

11.5 Magnetic flux

This relationship expresses that the field with the magnetic flux density B has no sources. This is synonymous with the assertion that the field lines are inherently self-contained.

$$\oint \vec{B} \cdot d\vec{A} = 0$$

11.6 Material equations

Electrostatic field: $\vec{D} = \varepsilon \cdot \vec{E}$

Electric current field: $\vec{J} = \kappa \cdot \vec{E}$

Magnetic field: $\vec{B} = \mu \cdot \vec{H}$

11.7 Summary

Maxwell's equations in integral form

Electrostatic field	Electric flow field	Magnetic field
$\oint \vec{D} \cdot d\vec{A} = \int \rho \cdot dV$ Gauss's law	$\oint \vec{J} \cdot d\vec{A} = -\frac{d}{dt} \int \rho \cdot dV$ Continuity equation	$\oint \vec{B} \cdot d\vec{A} = 0$ Magnetic flux
$\oint \vec{E} \cdot d\vec{s} = -\frac{d}{dt} \int \vec{B} \cdot d\vec{A}$ Law of induction		$\oint \vec{H} \cdot d\vec{s} = \int \left(\vec{J} + \frac{d\vec{D}}{dt} \right) \cdot d\vec{A}$ Ampere's law
$\vec{D} = \varepsilon \cdot \vec{E}$	$\vec{J} = \kappa \cdot \vec{E}$	$\vec{B} = \mu \cdot \vec{H}$
The electrostatic field is a source field.	The electric current field has sources when it is transient.	The magnetic field is source-free.
The electric field has vortices when a transient magnetic field is present.		The magnetic field is a vortex field.

Special case (e.g. vacuum):

space charge density	$\rho = 0$
conductivity	$\kappa = 0$
current density	$J = 0$

$\oint \vec{D} \cdot d\vec{A} = 0$	$\oint \vec{B} \cdot d\vec{A} = 0$	source-free
$\oint \vec{E} \cdot d\vec{s} = -\frac{d}{dt} \int \vec{B} \cdot d\vec{A}$	$\oint \vec{H} \cdot d\vec{s} = \frac{d}{dt} \int \vec{D} \cdot d\vec{A}$	with vortices
$\oint \vec{E} \cdot d\vec{s} = -\mu_0 \cdot \frac{d}{dt} \int \vec{H} \cdot d\vec{A}$	$\oint \vec{H} \cdot d\vec{s} = \varepsilon_0 \cdot \frac{d}{dt} \int_A \vec{E} \cdot d\vec{A}$	

Two time-variable source-free vortex fields where each one generates the other - **electromagnetic waves**

12. Current displacement (Skin-Effect)

12.1 Basics

A direct current *I* spreads evenly over the conductor cross section *A*. The current density is constant throughout straight conductors with location-independent cross-section. Furthermore, each current path is surrounded by a magnetic field *H*, which can be described according to Ampere's law:

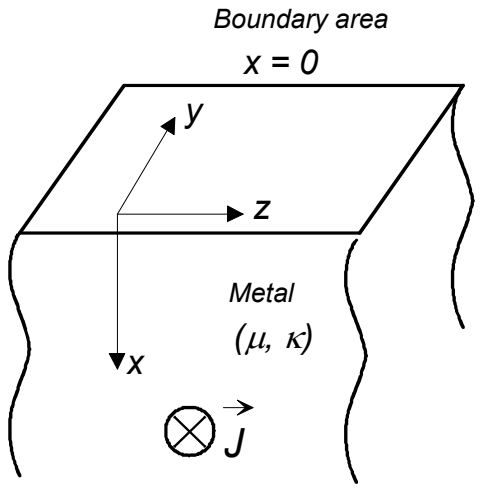
$$\oint \vec{H} \cdot d\vec{s} = \int \vec{J} \cdot d\vec{A}$$

If the current changes with time, the magnetic field changes also. As per the law of induction, additional electric field strengths are generated which superimpose the existing fields.

$$\oint \vec{E} \cdot d\vec{s} = -\frac{d}{dt} \int \vec{B} \cdot d\vec{A}$$

Due to this induction of additional electric fields, a weakening of the current density inside the conductor can be observed with alternating currents; the current is concentrated near the surface (skin effect).

The calculation of this property is relatively complex with round conductors. It is easier to determine the field distribution in the proximity of a plane surface.

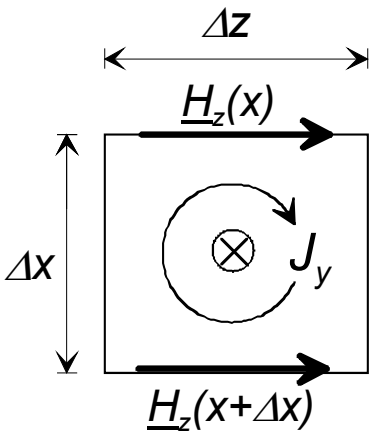


We examine a section from a metal block whose surface is located at $x = 0$. The current flows in the *y*-direction. The current density J_y is independent from *y* and *z*. For a transient process, J_y will be strongest at $x = 0$, and with increasing depth *x,* it will became smaller.

All field quantities to be considered *J*, *E* and *H* shall be assumed as sinusoidal time functions so that complex notations can be made:

$$\underline{J}_y(x,t) = \underline{J}_y(x) \cdot e^{j\omega t}$$
$$\underline{E}_y(x,t) = \underline{E}_y(x) \cdot e^{j\omega t}$$
$$\underline{H}_z(x,t) = \underline{H}_z(x) \cdot e^{j\omega t}$$

We now want to determine the depth dependence of the three field quantities. Therefore we must first review Ampere's law.



For an area element in each depth x must be deemed: $\oint \vec{H} \cdot d\vec{s} = \int \vec{J} \cdot d\vec{A}$

$$\underline{H}_z(x) \cdot \Delta z - \underline{H}_z(x+\Delta x) \cdot \Delta z \approx \underline{J}_y\left(x + \Delta x/2\right) \cdot \Delta x \cdot \Delta z$$

This ensues in:

$$\underline{J}_y\left(x + \Delta x/2\right) \approx -\frac{\underline{H}_z(x+\Delta x) - \underline{H}_z(x)}{\Delta x}$$

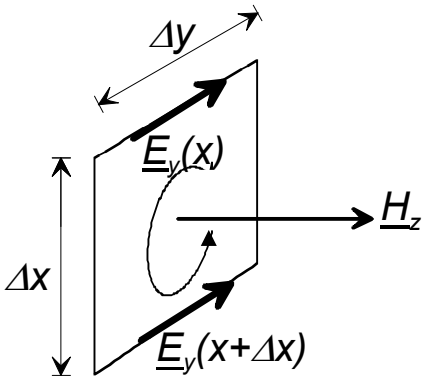
and for any small dimensions Δx follows:

$$\underline{J}_y(x) = -\frac{d\underline{H}_z(x)}{dx}$$

If the material equation $\vec{J} = \kappa \cdot \vec{E}$ is also applied, we obtain:

$$\frac{d\underline{H}_z(x)}{dx} = -\kappa \cdot \underline{E}_y(x)$$

Furthermore, the law of induction is valid as well.



For any locations x it must be deemed:

$$\oint \vec{E} \cdot d\vec{s} = -\frac{d}{dt} \int \vec{B} \cdot d\vec{A}$$

$$\underline{E}_y(x+\Delta x) \cdot \Delta y - \underline{E}_y(x) \cdot \Delta y \approx -j\omega \cdot \mu_0 \mu_r \cdot \underline{H}_z\left(x + \Delta x/2\right) \cdot \Delta x \cdot \Delta y$$

This results in:

$$\frac{\underline{E}_y(x+\Delta x) - \underline{E}_Y(x)}{\Delta x} \approx -j\omega \cdot \mu_0 \mu_r \cdot \underline{H}_z\left(x + \Delta x/2\right)$$

and for any small dimensions Δx follows:

$$\frac{d\underline{E}_y(x)}{dx} = -j\omega \cdot \mu_0 \mu_r \cdot \underline{H}_z(x)$$

This gives two interconnected first-order differential equations. To describe the field quantities $\underline{H}_z$ or $\underline{E}_y$ as a function of the depth *x* it can be done as follows:

The equation obtained with help of the law of induction becomes, by differentiating with respect to *x*:

$$\frac{d^2\underline{E}_y(x)}{dx^2} = -j\omega \cdot \mu_0 \mu_r \cdot \frac{d\underline{H}_z(x)}{dx}$$

If the relationship obtained by Ampere's law is inserted on the right side, the result is:

$$\frac{d^2\underline{E}_y(x)}{dx^2} = j\omega \cdot \mu_0 \mu_r \cdot \kappa \cdot \underline{E}_y(x)$$

Analogue, one gets:

$$\frac{d^2\underline{H}_z(x)}{dx^2} = j\omega \cdot \mu_0 \mu_r \cdot \kappa \cdot \underline{H}_z(x)$$

Since both equations correlate, it is sufficient for the moment to examine only one of the second-order differential equations. We do this for $\underline{E}_y(x)$.

The approach to a solution reads:

$$\underline{E}_y(x) = \underline{E}_1 \cdot e^{-\underline{\gamma} \cdot x} + \underline{E}_2 \cdot e^{+\underline{\gamma} \cdot x}$$

By differentiating twice, one obtains:

$$\frac{d\underline{E}_y(x)}{dx} = -\underline{\gamma}\cdot\underline{E}_1\cdot e^{-\underline{\gamma}\cdot x} + \underline{\gamma}\cdot\underline{E}_2\cdot e^{+\underline{\gamma}\cdot x} \qquad \frac{d^2\underline{E}_y(x)}{dx^2} = \underline{\gamma}^2\cdot\underline{E}_1\cdot e^{-\underline{\gamma}\cdot x} + \underline{\gamma}^2\cdot\underline{E}_2\cdot e^{+\underline{\gamma}\cdot x} = \underline{\gamma}^2\cdot\underline{E}_y(x)$$

By comparison with the differential equations for $\underline{E}_y(x)$ it follows: $\underline{\gamma}^2 = j\omega\cdot\mu_0\mu_r\cdot\kappa$

resp. $\underline{\gamma} = \sqrt{\omega\cdot\mu_0\mu_r\cdot\kappa}\cdot e^{j\pi/4} = \sqrt{\omega\cdot\mu_0\mu_r\cdot\kappa}\cdot\frac{(1+j)}{\sqrt{2}} = \sqrt{\frac{\omega\cdot\mu_0\mu_r\cdot\kappa}{2}}\cdot(1+j)$

For increasing depth x in the conductor, the electric field strength $\underline{E}_y(x)$ can in no way increase, instead, it must even decrease. This condition is only met by the term $\underline{E}_1$, so $\underline{E}_2$ must be zero. Then $\underline{E}_1$ is none other than the electric field strength on the surface of the conductor at $x = 0$.

For the electric field strength, this results in: $\underline{E}_y(x) = \underline{E}_y(0)\cdot e^{-\underline{\gamma}\cdot x}$

The magnetic field strength is analogous, resulting in: $\underline{H}_z(x) = \underline{H}_z(0)\cdot e^{-\underline{\gamma}\cdot x}$

$\underline{E}_y(0)$ and $\underline{H}_z(0)$ do not have the same phase shift (see 12.3.4). Indeed, $\underline{E}_y(t)$ and $\underline{H}_z(t)$ each have sinusoidal time-dependence at one location x, but are phase-shifted against each other.

12.2 Local and time dependence of the electric field strength

Next, we will be examining the electric field strength and its dependence not only on the depth x but also on the time t, or $\underline{E}_y(x, t)$. The real portion of the complex quantity $\underline{E}_y(x, t)$ emerges each time, therefore it is deemed:

$$E_y(x,t) = \mathrm{Re}\left\{\underline{E}_y(0)\cdot e^{-\sqrt{\frac{\omega\cdot\mu_0\mu_r\cdot\kappa}{2}}\cdot x}\cdot e^{-j\sqrt{\frac{\omega\cdot\mu_0\mu_r\cdot\kappa}{2}}\cdot x}\cdot e^{j\omega t}\right\}$$

The factor $\sqrt{\frac{\omega\cdot\mu_0\mu_r\cdot\kappa}{2}}$ describes the amplitude decrease of E as well as a phase rotation. If defined as $\delta = \sqrt{\frac{2}{\omega\cdot\mu_0\mu_r\cdot\kappa}}$, the above equation can be more easily formulated:

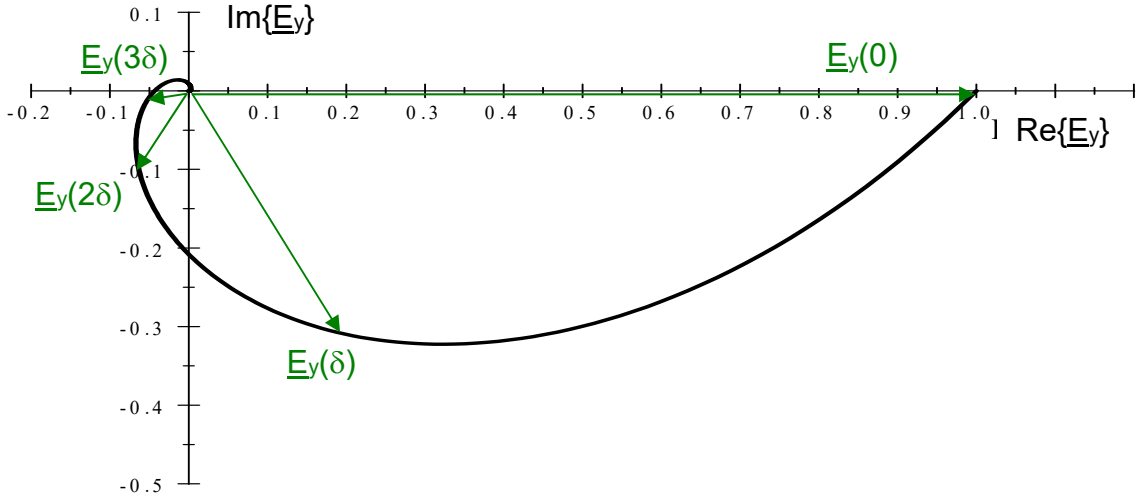
$$E_y(x,t) = \mathrm{Re}\left\{\underline{E}_y(0)\cdot e^{-\frac{x}{\delta}}\cdot e^{-j\frac{x}{\delta}}\cdot e^{j\omega t}\right\}$$

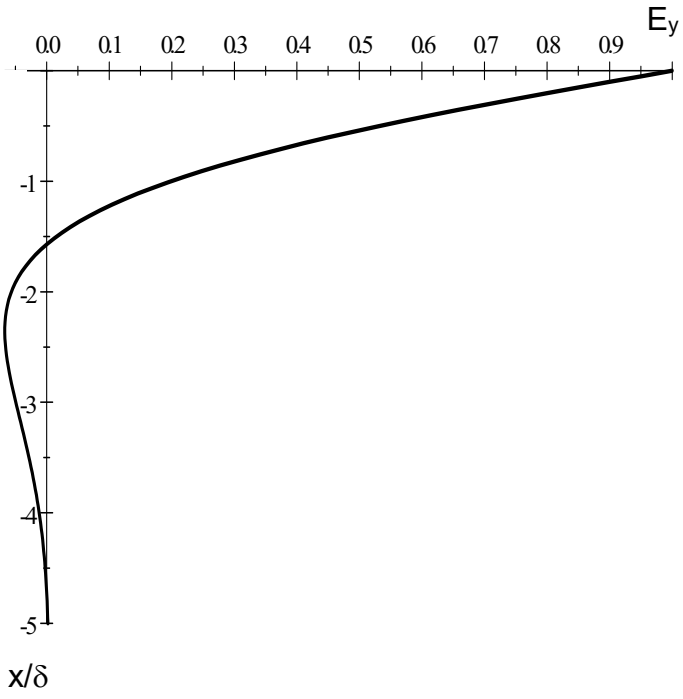
The quantity δ is identified as a penetration measure, alternatively penetration depth or skin depth. Upon reaching the depth $x = \delta$, the amount of the electric field strength has fallen to the value e^{-1}.

When the time dependence is set in such a way that $\underline{E}_y(0)$ is purely real and positive, so at $t = 0$ the electric field strength on the surface has its maximal value, the x-

dependence can be described by a locus from which for each depth *x* the amplitude and phase position relative to the value on the surface can be deduced:

$$\underline{E}_y(x,t=0) = E_y(0)\cdot e^{-\frac{x}{\delta}}\cdot e^{-j\frac{x}{\delta}}$$





The real field strengths are presented in the diagram for *t* = 0. They result as real portions again from the above locus.

It is recognizable that at the time *t* = 0, the electric field strength and, according to $\vec{J} = \kappa \cdot \vec{E}$, also the current density, not only decrease exponentially at growing depths *x*, but at certain depths even counter currents and counter field strengths appear in regard to the surface quantities.

In conclusion, all three field quantities will be described here in dependence of location and time. The following relationships ensue:

$$\underline{E}_y(x,t) = \underline{E}_y(0)\cdot e^{-x/\delta}\cdot e^{-jx/\delta}\cdot e^{j\omega t}$$

$$\underline{J}_y(x,t) = \underline{J}_y(0)\cdot e^{-x/\delta}\cdot e^{-jx/\delta}\cdot e^{j\omega t}$$

$$\underline{H}_z(x,t) = \underline{H}_z(0)\cdot e^{-x/\delta}\cdot e^{-jx/\delta}\cdot e^{j\omega t}$$

12.3 Technical relevance, Examples

12.3.1 Impedance of a part of a conductor

The effective resistance of a portion of a conductor is calculated from the length *l*, the width *b*, the depth *d* and the conductivity κ with location-independent current density *J* (direct current) to:

$$R = \frac{l}{\kappa \cdot b \cdot d} = \frac{U}{I} = \frac{E \cdot l}{J \cdot b \cdot d}$$

For the impedance with alternating current ensues: $\underline{Z} = \frac{\underline{U}}{\underline{I}}$

On a conductor portion the voltage is normally measured at the surface:

$$\underline{U} = \underline{E}(x=0) \cdot l = \underline{E}_y(0) \cdot l$$

The alternating current $\underline{I}$, in contrast, results from integrating the depth-dependent current density $\underline{J(x)}$ over the entire cross-section:

$$\underline{I} = b \cdot \int_{x=0}^{\infty} \underline{J}(x) dx$$

With $\underline{J}_y(x) = \kappa \cdot \underline{E}_y(x) = \kappa \cdot \underline{E}_y(0) \cdot e^{-(1+j) \cdot x/\delta}$ the result is:

$$\underline{I} = b \cdot \kappa \cdot \underline{E}_y(0) \int_{x=0}^{\infty} e^{-(1+j) \cdot x/\delta} dx = \frac{b \cdot \kappa \cdot \underline{E}_y(0) \cdot \delta}{1+j} = \frac{b \cdot \kappa \cdot \underline{E}_y(0) \cdot \delta}{2} \cdot (1-j)$$

Therefore the impedance results in:

$$\underline{Z} = \frac{\underline{U}}{\underline{I}} = \frac{\underline{E_y(0)} \cdot l}{b \cdot \kappa \cdot \underline{E}_y(0) \cdot \delta}(1+j) = \frac{l}{b \cdot \kappa \cdot \delta}(1+j) = R + jX_i \quad \text{and} \quad R = \frac{l}{b \cdot \kappa \cdot \delta}$$

The effective resistance *R* of a conductor portion with infinite depth (respectively with very distinct skin effect) is therefore just as large as the resistance of a conducting portion of the depth δ at which the current density is evenly distributed throughout. This "equivalent skin depth" δ is both material dependent and frequency dependent.

Furthermore, the imaginary portion of the impedance (reactance) is exactly as large as the effective resistance with distinct skin effect. This imaginary portion X_i represents the so-called inner inductance.

12.3.2 Losses in case of distinct skin effect

With the effective resistance (real portion of the impedance) which has been calculated in 12.3.1 and the current *I*, the power loss for a conductor portion can be calculated. This can be done here with two formulations.

The 1st solution can be attained by integration in a conductor portion with distinct skin effect. It is deemed:

$$\overline{P} = \frac{l \cdot b}{2} \cdot \int_{x=0}^{\infty} \underline{E}_y(x) \cdot \underline{J}_y^*(x) \cdot dx$$

Note: For the statement of $\underline{J}_y^*(x)$ see complex power calculation

$$\overline{P} = \frac{l \cdot b}{2 \cdot \kappa} \cdot \int_{x=0}^{\infty} J_y^2(x=0) \cdot e^{-\frac{2x}{\delta}} \cdot dx = \frac{l \cdot b \cdot J_y^2(x=0)}{2 \cdot \kappa} \left(-\frac{\delta}{2}\right) \left[e^{-\frac{2x}{\delta}} \right]_{x=0}^{\infty} = \frac{l \cdot b \cdot \delta \cdot J_y^2(x=0)}{4 \cdot \kappa}$$

As a trial, we will verify that the result is achievable with the formula $\overline{P} = \frac{1}{2} \cdot \hat{I}^2 \cdot R$ (2nd solution). From 12.3.1. it follows:

$$\hat{I} = \frac{b \cdot \kappa \cdot \underline{E}_y(0) \cdot \delta}{\sqrt{2}} = \frac{b \cdot \underline{J}_y(0) \cdot \delta}{\sqrt{2}} \quad \text{and} \quad R = \frac{l}{b \cdot \kappa \cdot \delta}.$$

Therefrom, it is obtained: $\overline{P} = \frac{1}{2} \cdot \frac{b^2 \cdot \delta^2 \cdot J_y^2(x=0) \cdot l}{2 \cdot b \cdot \kappa \cdot \delta} = \frac{l \cdot b \cdot \delta \cdot J_y^2(x=0)}{4 \cdot \kappa}$

12.3.3 Skin depth

With distinct skin effect, it can be assumed that the current only flows in a thin layer on the surface with the depth δ.

$$\delta = \sqrt{\frac{2}{\omega \cdot \mu_0 \mu_r \cdot \kappa}}$$

For three selected materials (metals) the following skin depths occur:

frequency	copper	aluminium	steel
	$\mu_R = 1$	$\mu_r = 1$	$\mu_r = 200$
	$\kappa = 57.14$ Sm/mm²	$\kappa = 34.8$ Sm/mm²	$\kappa = 7.69$ Sm/mm²
	δ in mm	δ in mm	δ in mm
1 Hz	66.6	85.3	12.8
10 Hz	21.1	27.0	4.06
50 Hz	9.42	12.1	1.81
100 Kz	6.66	8.53	1.28
1 kHz	2.11	2.70	0.406
10 kHz	0.666	0.853	0.128
100 kHz	0.211	0.270	0.0406
1 MHz	0.0666	0.0853	0.0128
10 MHz	0.0211	0.0270	0.00406
100 MHz	0.00666	0.00853	0.00128
1 GHz	0.00211	0.00270	0.000406
10 GHz	0.000666	0.000853	0.000128

As approximation formulas for resistance with distinct skin effect, the following can be applied:

- for round conductors: $R = \frac{l}{\kappa \cdot d \cdot \pi \cdot \delta}$
- for cornered conductors: $R = \frac{l}{\kappa \cdot 2 \cdot (b + d) \cdot \delta}$

12.3.4 Magnetic field strength

As shown in the depiction in section 12.2, the electric field strength $\underline{E}_y(0)$ is the foremost of all field strengths, rushing over the surface to the inside of the conductor. Because of $\vec{J} = \kappa \cdot \vec{E}$, this must naturally also be valid for the current density $\underline{J}_y$.

The total current $\underline{I}$ results from the integration of all current densities in the area. Thereby the total current $\underline{I}$ must rush after the current density on the surface $\underline{J}_y(0)$. After the voltage $\underline{U}$ is measured on the surface, and it results from the field strength $\underline{E}_y(0)$ on the surface, the voltage $\underline{U}$ must rush ahead of the current $\underline{I}$. The phase angle can be determined using two methods.

When the relationship $\underline{Z} = R + jX_i$ is inserted into the equation $\underline{U} = \underline{Z} \cdot \underline{I}$ according to 12.3.1, whereby in the case of a distinct skin effect, $R = X_i$ applies, it follows that the voltage *U* rushes ahead of the current *I* at 45°. This phase angle must also apply to the magnetic field strength $\underline{H}_z(0)$ on the surface because it includes the total current (Ampere's law !). Consequently, the magnetic field strength $\underline{H}_z(0)$ must rush after the electric field strength $\underline{E}_y(0)$ on the surface at a 45°angle.

This correlation can also be achieved using the basic equations depicted in 12.1. From

$$\frac{d\underline{E}_y(x)}{dx} = -j\omega \cdot \mu_0\mu_r \cdot \underline{H}_z(x) \qquad \text{with} \qquad \underline{E}_y(x) = \underline{E}_y(0) \cdot e^{-\underline{\gamma} \cdot x} \qquad \underline{H}_z(x) = \underline{H}_z(0) \cdot e^{-\underline{\gamma} \cdot x}$$

follows: $-\underline{\gamma} \cdot \underline{E}_y(0) \cdot e^{-\underline{\gamma} \cdot x} = -j\omega \cdot \mu_0\mu_r \cdot \underline{H}_z(0) \cdot e^{-\underline{\gamma} \cdot x}$ and $\underline{H}_z(0) = \frac{\underline{\gamma}}{j\omega \cdot \mu_0\mu_r} \cdot \underline{E}_y(0)$.

With the quantity $\underline{\gamma} = \sqrt{\omega \cdot \mu_0\mu_r \cdot \kappa} \cdot e^{j\pi/4}$ according to 12.1, follows:

$$\underline{H}_z(0) = \frac{\sqrt{\omega \cdot \mu_0\mu_r \cdot \kappa} \cdot e^{j\pi/4}}{e^{j\pi/2} \cdot \omega \cdot \mu_0\mu_r} \cdot \underline{E}_y(0) = \sqrt{\frac{\kappa}{\omega \cdot \mu_0\mu_r}} \cdot e^{-j\pi/4} \cdot \underline{E}_y(0).$$